JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Lattice of Congruences in Many Sorted Algebra

Robert Milewski
Warsaw University
Białystok

MML Identifier: MSUALG_5.

WWW: http://mizar.org/JFM/Vol8/msualg_5.html

The articles [14], [8], [18], [19], [21], [5], [7], [6], [20], [13], [4], [15], [16], [10], [11], [22], [9], [2], [12], [3], [17], and [1] provide the notation and terminology for this paper.

1. More on Equivalence Relations

For simplicity, we adopt the following convention: I, X are sets, M is a many sorted set indexed by I, R_1 is a binary relation on X, and E_1, E_2, E_3 are equivalence relations of X.

Next we state the proposition

(1) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

Let X be a set and let R be a binary relation on X. The functor $\mathrm{EqCl}(R)$ yields an equivalence relation of X and is defined by:

(Def. 1) $R \subseteq \mathrm{EqCl}(R)$ and for every equivalence relation E_2 of X such that $R \subseteq E_2$ holds $\mathrm{EqCl}(R) \subseteq E_2$.

Next we state three propositions:

(2) $E_1 \sqcup E_2 = \mathrm{EqCl}(E_1 \cup E_2)$.

(3) $\mathrm{EqCl}(E_1) = E_1$.

(4) $\nabla_X \cup R_1 = \nabla_X$.

2. Lattice of Equivalence Relations

Let X be a set. The functor $\mathrm{EqRelLatt}(X)$ yielding a strict lattice is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $\mathrm{EqRelLatt}(X) = \{x; x$ ranges over relations between X and X: x is an equivalence relation of $X\}$, and

(ii) for all equivalence relations x, y of X holds (the meet operation of $\mathrm{EqRelLatt}(X))(x, y) = x \cap y$ and (the join operation of $\mathrm{EqRelLatt}(X))(x, y) = x \sqcup y$.

3. Many Sorted Equivalence Relations

Let us consider I, M. Note that there exists a many sorted relation indexed by M which is equivalence.

Let us consider I, M. An equivalence relation of M is an equivalence many sorted relation indexed by M.

We adopt the following convention: I denotes a non empty set, M denotes a many sorted set indexed by I, and E_4, E_1, E_2, E_3 denote equivalence relations of M.

Let I be a non empty set, let M be a many sorted set indexed by I, and let R be a many sorted relation indexed by M. The functor $\mathrm{EqCl}(R)$ yields an equivalence relation of M and is defined by:

(Def. 3) For every element i of I holds $(\mathrm{EqCl}(R))(i) = \mathrm{EqCl}(R(i))$.

One can prove the following proposition

(5) $\mathrm{EqCl}(E_4) = E_4$.

4. Lattice of Many Sorted Equivalence Relations

Let I be a non empty set, let M be a many sorted set indexed by I, and let E_1, E_2 be equivalence relations of M. The functor $E_1 \sqcup E_2$ yielding an equivalence relation of M is defined by:

(Def. 4) There exists a many sorted relation E_3 indexed by M such that $E_3 = E_1 \cup E_2$ and $E_1 \sqcup E_2 = \mathrm{EqCl}(E_3)$.

Let us notice that the functor $E_1 \sqcup E_2$ is commutative.

One can prove the following propositions:

(6) $E_1 \cup E_2 \subseteq E_1 \sqcup E_2$.

(7) For every equivalence relation E_4 of M such that $E_1 \cup E_2 \subseteq E_4$ holds $E_1 \sqcup E_2 \subseteq E_4$.

(8) If $E_1 \cup E_2 \subseteq E_3$ and for every equivalence relation E_4 of M such that $E_1 \cup E_2 \subseteq E_4$ holds $E_3 \subseteq E_4$, then $E_3 = E_1 \sqcup E_2$.

(9) $E_4 \sqcup E_4 = E_4$.

(10) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

(11) $E_1 \cap (E_1 \sqcup E_2) = E_1$.

(12) For every equivalence relation E_4 of M such that $E_4 = E_1 \cap E_2$ holds $E_1 \sqcup E_4 = E_1$.

(13) For all equivalence relations E_1, E_2 of M holds $E_1 \cap E_2$ is an equivalence relation of M.

Let I be a non empty set and let M be a many sorted set indexed by I. The functor $\mathrm{EqRelLatt}(M)$ yields a strict lattice and is defined by the conditions (Def. 5).

(Def. 5)(i) For every set x holds $x \in$ the carrier of $\mathrm{EqRelLatt}(M)$ iff x is an equivalence relation of M, and

(ii) for all equivalence relations x, y of M holds (the meet operation of $\mathrm{EqRelLatt}(M))(x, y) = x \cap y$ and (the join operation of $\mathrm{EqRelLatt}(M))(x, y) = x \sqcup y$.

5. LATTICE OF CONGRUENCES IN MANY SORTED ALGEBRA

Let S be a non empty many sorted signature and let A be an algebra over S. Observe that every many sorted relation indexed by A which is equivalence is also equivalence.

In the sequel S denotes a non void non empty many sorted signature and A denotes a non-empty algebra over S.

Next we state several propositions:

(14) Let o be an operation symbol of S, C_1, C_2 be congruences of A, x_1, y_1 be sets, and a_1, b_1 be finite sequences. Suppose $\langle x_1, y_1 \rangle \in C_1(\mathrm{Arity}(o)_{\mathrm{len}\, a_1+1}) \cup C_2(\mathrm{Arity}(o)_{\mathrm{len}\, a_1+1})$. Let x, y be elements of $\mathrm{Args}(o,A)$. Suppose $x = a_1 \frown \langle x_1 \rangle \frown b_1$ and $y = a_1 \frown \langle y_1 \rangle \frown b_1$. Then $\langle (\mathrm{Den}(o, A))(x), (\mathrm{Den}(o,A))(y) \rangle \in C_1(\text{the result sort of } o) \cup C_2(\text{the result sort of } o)$.

(15) Let o be an operation symbol of S, C_1, C_2 be congruences of A, and C be an equivalence many sorted relation indexed by A. Suppose $C = C_1 \sqcup C_2$. Let x_1, y_1 be sets, n be a natural number, and a_1, a_2, b_1 be finite sequences. Suppose $\mathrm{len}\, a_1 = n$ and $\mathrm{len}\, a_1 = \mathrm{len}\, a_2$ and for every natural number k such that $k \in \mathrm{dom}\, a_1$ holds $\langle a_1(k), a_2(k) \rangle \in C(\mathrm{Arity}(o)_k)$. Suppose $\langle (\mathrm{Den}(o,A))(a_1 \frown \langle x_1 \rangle \frown b_1), (\mathrm{Den}(o,A))(a_2 \frown \langle x_1 \rangle \frown b_1) \rangle \in C(\text{the result sort of } o)$ and $\langle x_1, y_1 \rangle \in C(\mathrm{Arity}(o)_{n+1})$. Let x be an element of $\mathrm{Args}(o,A)$. If $x = a_1 \frown \langle x_1 \rangle \frown b_1$, then $\langle (\mathrm{Den}(o, A))(x), (\mathrm{Den}(o,A))(a_2 \frown \langle y_1 \rangle \frown b_1) \rangle \in C(\text{the result sort of } o)$.

(16) Let o be an operation symbol of S, C_1, C_2 be congruences of A, and C be an equivalence many sorted relation indexed by A. Suppose $C = C_1 \sqcup C_2$. Let x, y be elements of $\mathrm{Args}(o,A)$. Suppose that for every natural number n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n) \rangle \in C(\mathrm{Arity}(o)_n)$. Then $\langle (\mathrm{Den}(o,A))(x), (\mathrm{Den}(o,A))(y) \rangle \in C(\text{the result sort of } o)$.

(17) For all congruences C_1, C_2 of A holds $C_1 \sqcup C_2$ is a congruence of A.

(18) For all congruences C_1, C_2 of A holds $C_1 \cap C_2$ is a congruence of A.

Let us consider S and let A be a non-empty algebra over S. The functor $\mathrm{CongrLatt}(A)$ yields a strict sublattice of $\mathrm{EqRelLatt}(\text{the sorts of } A)$ and is defined as follows:

(Def. 6) For every set x holds $x \in$ the carrier of $\mathrm{CongrLatt}(A)$ iff x is a congruence of A.

We now state two propositions:

(19) $\mathrm{id}_{\text{the sorts of } A}$ is a congruence of A.

(20) $[\![\text{the sorts of } A, \text{the sorts of } A]\!]$ is a congruence of A.

Let us consider S, A. Observe that $\mathrm{CongrLatt}(A)$ is bounded.

We now state two propositions:

(21) $\bot_{\mathrm{CongrLatt}(A)} = \mathrm{id}_{\text{the sorts of } A}$.

(22) $\top_{\mathrm{CongrLatt}(A)} = [\![\text{the sorts of } A, \text{the sorts of } A]\!]$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[7] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/nat_lat.html.

[10] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[11] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_4.html.

[12] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pralg_2.html.

[13] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/eqrel_1.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[16] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[17] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[20] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[21] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

[22] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

Received January 11, 1996

Published January 2, 2004