JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Many Sorted Quotient Algebra

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. This article introduces the construction of a many sorted quotient algebra. A few preliminary notions such as a many sorted relation, a many sorted equivalence relation, a many sorted congruence and the set of all classes of a many sorted relation are also formulated.

MML Identifier: MSUALG_4.

WWW: http://mizar.org/JFM/Vol6/msualg_4.html

The articles [10], [14], [15], [3], [16], [5], [9], [4], [2], [1], [13], [11], [7], [12], [6], and [8] provide the notation and terminology for this paper.

1. Many Sorted Relation

In this paper S denotes a non void non empty many sorted signature, U_1 denotes an algebra over S, o denotes an operation symbol of S, and s denotes a sort symbol of S.

Let I_1 be a function. We say that I_1 is binary relation yielding if and only if:

(Def. 1) For every set x such that $x \in \operatorname{dom} I_1$ holds $I_1(x)$ is a binary relation.

Let I be a set. Observe that there exists a many sorted set indexed by I which is binary relation yielding.

Let I be a set. A many sorted relation indexed by I is a binary relation yielding many sorted set indexed by I.

Let I be a set and let A, B be many sorted sets indexed by I. A many sorted set indexed by I is said to be a many sorted relation between A and B if:

(Def. 2) For every set i such that $i \in I$ holds $\mathrm{it}(i)$ is a relation between $A(i)$ and $B(i)$.

Let I be a set and let A, B be many sorted sets indexed by I. Observe that every many sorted relation between A and B is binary relation yielding.

Let I be a set and let A be a many sorted set indexed by I. A many sorted relation indexed by A is a many sorted relation between A and A.

Let I be a set, let A be a many sorted set indexed by I, and let I_1 be a many sorted relation indexed by A. We say that I_1 is equivalence if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let i be a set and R be a binary relation on $A(i)$. If $i \in I$ and $I_1(i) = R$, then R is an equivalence relation of $A(i)$.

Let I be a non empty set, let A, B be many sorted sets indexed by I, let F be a many sorted relation between A and B, and let i be an element of I. Then $F(i)$ is a relation between $A(i)$ and $B(i)$.

Let S be a non empty many sorted signature and let U_1 be an algebra over S. A many sorted relation indexed by U_1 is a many sorted relation indexed by the sorts of U_1.

Let S be a non empty many sorted signature, let U_1 be an algebra over S, and let I_1 be a many sorted relation indexed by U_1. We say that I_1 is equivalence if and only if:

(Def. 5)[1] I_1 is equivalence.

Let S be a non void non empty many sorted signature and let U_1 be an algebra over S. One can check that there exists a many sorted relation indexed by U_1 which is equivalence.

The following proposition is true

(1) Let R be an equivalence many sorted relation indexed by U_1. Then $R(s)$ is an equivalence relation of (the sorts of $U_1)(s)$.

Let S be a non void non empty many sorted signature, let U_1 be a non-empty algebra over S, and let R be an equivalence many sorted relation indexed by U_1. We say that R is MSCongruence-like if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let o be an operation symbol of S and x, y be elements of $\mathrm{Args}(o,U_1)$. Suppose that for every natural number n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n)\rangle \in R(\mathrm{Arity}(o)_n)$. Then $\langle(\mathrm{Den}(o, U_1))(x), (\mathrm{Den}(o,U_1))(y)\rangle \in R$(the result sort of o).

Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. One can check that there exists an equivalence many sorted relation indexed by U_1 which is MSCongruence-like.

Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. A congruence of U_1 is a MSCongruence-like equivalence many sorted relation indexed by U_1.

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation indexed by U_1, and let i be an element of S. Then $R(i)$ is an equivalence relation of (the sorts of $U_1)(i)$.

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation indexed by U_1, let i be an element of S, and let x be an element of (the sorts of $U_1)(i)$. The functor $[x]_R$ yielding a subset of (the sorts of $U_1)(i)$ is defined as follows:

(Def. 7) $[x]_R = [x]_{R(i)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{Classes}\, R$ yielding a non-empty many sorted set indexed by the carrier of S is defined by:

(Def. 8) For every element s of S holds $(\mathrm{Classes}\, R)(s) = \mathrm{Classes}\, R(s)$.

2. Many Sorted Quotient Algebra

Let us consider S, let M_1, M_2 be many sorted sets indexed by the operation symbols of S, let F be a many sorted function from M_1 into M_2, and let o be an operation symbol of S. Then $F(o)$ is a function from $M_1(o)$ into $M_2(o)$.

Let I be a non empty set, let p be a finite sequence of elements of I, and let X be a non-empty many sorted set indexed by I. Then $X \cdot p$ is a non-empty many sorted set indexed by $\mathrm{dom}\, p$.

Let us consider S, o, let A be a non-empty algebra over S, let R be a congruence of A, and let x be an element of $\mathrm{Args}(o,A)$. The functor $R\#x$ yields an element of $\prod(\mathrm{Classes}\, R \cdot \mathrm{Arity}(o))$ and is defined by:

(Def. 9) For every natural number n such that $n \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $(R\#x)(n) = [x(n)]_{R(\mathrm{Arity}(o)_n)}$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R,o)$ yielding a function from ((the sorts of A) $\cdot$ (the result sort of S))(o) into ($\mathrm{Classes}\, R \cdot$ the result sort of $S)(o)$ is defined by:

(Def. 10) For every element x of (the sorts of A)(the result sort of o) holds $(\mathrm{QuotRes}(R,o))(x) = [x]_R$.

The functor $\mathrm{QuotArgs}(R,o)$ yields a function from ((the sorts of $A)^{\#} \cdot$ the arity of $S)(o)$ into $((\mathrm{Classes}\, R)^{\#} \cdot$ the arity of $S)(o)$ and is defined as follows:

(Def. 11) For every element x of $\mathrm{Args}(o,A)$ holds $(\mathrm{QuotArgs}(R,o))(x) = R\#x$.

[1] The definition (Def. 4) has been removed.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R)$ yielding a many sorted function from (the sorts of A) $\cdot$ (the result sort of S) into $\mathrm{Classes}\,R \cdot$ the result sort of S is defined as follows:

(Def. 12) For every operation symbol o of S holds $(\mathrm{QuotRes}(R))(o) = \mathrm{QuotRes}(R,o)$.

The functor $\mathrm{QuotArgs}(R)$ yields a many sorted function from (the sorts of A)$^{\#} \cdot$ the arity of S into $(\mathrm{Classes}\,R)^{\#} \cdot$ the arity of S and is defined by:

(Def. 13) For every operation symbol o of S holds $(\mathrm{QuotArgs}(R))(o) = \mathrm{QuotArgs}(R,o)$.

Next we state the proposition

(2) Let A be a non-empty algebra over S, R be a congruence of A, and x be a set. Suppose $x \in ((\mathrm{Classes}\,R)^{\#} \cdot$ the arity of $S)(o)$. Then there exists an element a of $\mathrm{Args}(o,A)$ such that $x = R\#a$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotCharact}(R,o)$ yields a function from $((\mathrm{Classes}\,R)^{\#} \cdot$ the arity of $S)(o)$ into $(\mathrm{Classes}\,R \cdot$ the result sort of $S)(o)$ and is defined by:

(Def. 14) For every element a of $\mathrm{Args}(o,A)$ such that $R\#a \in ((\mathrm{Classes}\,R)^{\#} \cdot$ the arity of $S)(o)$ holds $(\mathrm{QuotCharact}(R,o))(R\#a) = (\mathrm{QuotRes}(R,o) \cdot \mathrm{Den}(o,A))(a)$.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotCharact}(R)$ yields a many sorted function from $(\mathrm{Classes}\,R)^{\#} \cdot$ the arity of S into $\mathrm{Classes}\,R \cdot$ the result sort of S and is defined as follows:

(Def. 15) For every operation symbol o of S holds $(\mathrm{QuotCharact}(R))(o) = \mathrm{QuotCharact}(R,o)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor U_1/R yields an algebra over S and is defined as follows:

(Def. 16) $U_1/R = \langle \mathrm{Classes}\,R, \mathrm{QuotCharact}(R) \rangle$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. One can verify that U_1/R is strict and non-empty.

Let us consider S, let U_1 be a non-empty algebra over S, let R be a congruence of U_1, and let s be a sort symbol of S. The functor $\mathrm{MSNatHom}(U_1,R,s)$ yielding a function from (the sorts of U_1)(s) into $(\mathrm{Classes}\,R)(s)$ is defined by:

(Def. 17) For every set x such that $x \in$ (the sorts of U_1)(s) holds $(\mathrm{MSNatHom}(U_1,R,s))(x) = [x]_{R(s)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{MSNatHom}(U_1,R)$ yielding a many sorted function from U_1 into U_1/R is defined as follows:

(Def. 18) For every sort symbol s of S holds $(\mathrm{MSNatHom}(U_1,R))(s) = \mathrm{MSNatHom}(U_1,R,s)$.

One can prove the following proposition

(3) For every non-empty algebra U_1 over S and for every congruence R of U_1 holds $\mathrm{MSNatHom}(U_1,R)$ is an epimorphism of U_1 onto U_1/R.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. The functor $\mathrm{Congruence}(F,s)$ yielding an equivalence relation of (the sorts of U_1)(s) is defined by:

(Def. 19) For all elements x, y of (the sorts of U_1)(s) holds $\langle x, y \rangle \in \mathrm{Congruence}(F,s)$ iff $F(s)(x) = F(s)(y)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{Congruence}(F)$ yields a congruence of U_1 and is defined as follows:

(Def. 20) For every sort symbol s of S holds $(\mathrm{Congruence}(F))(s) = \mathrm{Congruence}(F, s)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{MSHomQuot}(F, s)$ yields a function from (the sorts of $U_1 / \mathrm{Congruence}(F))(s)$ into (the sorts of $U_2)(s)$ and is defined as follows:

(Def. 21) For every element x of (the sorts of $U_1)(s)$ holds $(\mathrm{MSHomQuot}(F, s))([x]_{\mathrm{Congruence}(F,s)}) = F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. The functor $\mathrm{MSHomQuot}(F)$ yields a many sorted function from $U_1 / \mathrm{Congruence}(F)$ into U_2 and is defined by:

(Def. 22) For every sort symbol s of S holds $(\mathrm{MSHomQuot}(F))(s) = \mathrm{MSHomQuot}(F, s)$.

We now state three propositions:

(4) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then $\mathrm{MSHomQuot}(F)$ is a monomorphism of $U_1 / \mathrm{Congruence}(F)$ into U_2.

(5) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2. Then $\mathrm{MSHomQuot}(F)$ is an isomorphism of $U_1 / \mathrm{Congruence}(F)$ and U_2.

(6) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. If F is an epimorphism of U_1 onto U_2, then $U_1 / \mathrm{Congruence}(F)$ and U_2 are isomorphic.

REFERENCES

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[6] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_3.html`.

[7] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[8] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/pralg_2.html`.

[9] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/eqrel_1.html`.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[11] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[12] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_1.html`.

[13] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[16] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received May 6, 1994

Published January 2, 2004