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The Correspondence Between Lattices of Subalgebras of Universal Algebras and Many Sorted Algebras

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Summary. The main goal of this paper is to show some properties of subalgebras of universal algebras and many sorted algebras, and then the isomorphic correspondence between lattices of such subalgebras.

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The articles [11], [15], [8], [17], [16], [5], [7], [6], [12], [10], [3], [2], [14], [1], [9], [13], and [4] provide the notation and terminology for this paper.

1. Preliminaries

In this paper a is a set and i is a natural number.

The following propositions are true:

(1) $(\square \longmapsto a)(0) = \emptyset$.

(2) $(\square \longmapsto a)(1) = \langle a \rangle$.

(3) $(\square \longmapsto a)(2) = \langle a, a \rangle$.

(4) $(\square \longmapsto a)(3) = \langle a, a, a \rangle$.

(5) For every finite sequence f of elements of $\{0\}$ holds $f = i \mapsto 0$ iff $\operatorname{len} f = i$.

(6) For every finite sequence f such that $f = (\square \longmapsto 0)(i)$ holds $\operatorname{len} f = i$.

2. Some Properties of Subalgebras of Universal and Many Sorted Algebras

Next we state the proposition

(7) For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds $\operatorname{MSSign}(U_1) = \operatorname{MSSign}(U_2)$.

Let U_0 be a universal algebra. One can check that the characteristic of U_0 is function yielding.

One can prove the following propositions:

(8) Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a subset of $\mathrm{MSAlg}(U_2)$. Suppose $B =$ the sorts of $\mathrm{MSAlg}(U_1)$. Let o be an operation symbol of $\mathrm{MSSign}(U_2)$ and a be an operation symbol of $\mathrm{MSSign}(U_1)$. If $a = o$, then $\mathrm{Den}(a, \mathrm{MSAlg}(U_1)) = \mathrm{Den}(o, \mathrm{MSAlg}(U_2)) \restriction \mathrm{Args}(a, \mathrm{MSAlg}(U_1))$.

(9) For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds the sorts of $\mathrm{MSAlg}(U_1)$ are a subset of $\mathrm{MSAlg}(U_2)$.

(10) Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a subset of $\mathrm{MSAlg}(U_2)$. If $B =$ the sorts of $\mathrm{MSAlg}(U_1)$, then B is operations closed.

(11) Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a subset of $\mathrm{MSAlg}(U_2)$. If $B =$ the sorts of $\mathrm{MSAlg}(U_1)$, then the characteristics of $\mathrm{MSAlg}(U_1) = \mathrm{Opers}(\mathrm{MSAlg}(U_2), B)$.

(12) For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds $\mathrm{MSAlg}(U_1)$ is a subalgebra of $\mathrm{MSAlg}(U_2)$.

(13) For all universal algebras U_1, U_2 such that $\mathrm{MSAlg}(U_1)$ is a subalgebra of $\mathrm{MSAlg}(U_2)$ holds the carrier of U_1 is a subset of U_2.

(14) Let U_1, U_2 be universal algebras. Suppose $\mathrm{MSAlg}(U_1)$ is a subalgebra of $\mathrm{MSAlg}(U_2)$. Let B be a non empty subset of U_2. If $B =$ the carrier of U_1, then B is operations closed.

(15) Let U_1, U_2 be universal algebras. Suppose $\mathrm{MSAlg}(U_1)$ is a subalgebra of $\mathrm{MSAlg}(U_2)$. Let B be a non empty subset of U_2. If $B =$ the carrier of U_1, then the characteristic of $U_1 = \mathrm{Opers}(U_2, B)$.

(16) For all universal algebras U_1, U_2 such that $\mathrm{MSAlg}(U_1)$ is a subalgebra of $\mathrm{MSAlg}(U_2)$ holds U_1 is a subalgebra of U_2.

In the sequel M_1 is a segmental trivial non void non empty many sorted signature and A is a non-empty algebra over M_1.

The following propositions are true:

(17) For every non-empty subalgebra B of A holds the carrier of $\mathrm{Alg}_{\mathbf{1}}(B)$ is a subset of $\mathrm{Alg}_{\mathbf{1}}(A)$.

(18) Let B be a non-empty subalgebra of A and S be a non empty subset of $\mathrm{Alg}_{\mathbf{1}}(A)$. If $S =$ the carrier of $\mathrm{Alg}_{\mathbf{1}}(B)$, then S is operations closed.

(19) Let B be a non-empty subalgebra of A and S be a non empty subset of $\mathrm{Alg}_{\mathbf{1}}(A)$. If $S =$ the carrier of $\mathrm{Alg}_{\mathbf{1}}(B)$, then the characteristic of $\mathrm{Alg}_{\mathbf{1}}(B) = \mathrm{Opers}(\mathrm{Alg}_{\mathbf{1}}(A), S)$.

(20) For every non-empty subalgebra B of A holds $\mathrm{Alg}_{\mathbf{1}}(B)$ is a subalgebra of $\mathrm{Alg}_{\mathbf{1}}(A)$.

(21) Let S be a non empty non void many sorted signature and A, B be algebras over S. Then A is a subalgebra of B if and only if A is a subalgebra of the algebra of B.

(22) For all universal algebras A, B holds $\mathrm{signature} A = \mathrm{signature} B$ iff $\mathrm{MSSign}(A) = \mathrm{MSSign}(B)$.

(23) Let A be a non-empty algebra over M_1. Suppose the carrier of $M_1 = \{0\}$. Then $\mathrm{MSSign}(\mathrm{Alg}_{\mathbf{1}}(A)) =$ the many sorted signature of M_1.

(24) Let A, B be non-empty algebras over M_1. Suppose the carrier of $M_1 = \{0\}$ and $\mathrm{Alg}_{\mathbf{1}}(A) = \mathrm{Alg}_{\mathbf{1}}(B)$. Then the algebra of $A =$ the algebra of B.

(25) Let A be a non-empty algebra over M_1. If the carrier of $M_1 = \{0\}$, then the sorts of $A =$ the sorts of $\mathrm{MSAlg}(\mathrm{Alg}_{\mathbf{1}}(A))$.

(26) For every non-empty algebra A over M_1 such that the carrier of $M_1 = \{0\}$ holds $\mathrm{MSAlg}(\mathrm{Alg}_{\mathbf{1}}(A)) =$ the algebra of A.

(27) Let A be a universal algebra and B be a strict non-empty subalgebra of $\mathrm{MSAlg}(A)$. If the carrier of $\mathrm{MSSign}(A) = \{0\}$, then $\mathrm{Alg}_{\mathbf{1}}(B)$ is a subalgebra of A.

3. The Correspondence Between Lattices of Subalgebras of Universal and Many Sorted Algebras

Next we state three propositions:

(28) Let A be a universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of $\mathrm{MSAlg}(A)$. Suppose $a_2 = \mathrm{MSAlg}(a_1)$ and $b_2 = \mathrm{MSAlg}(b_1)$. Then (the sorts of $a_2) \cup ($the sorts of $b_2) = 0 \dot{\longmapsto} (($the carrier of $a_1) \cup ($the carrier of $b_1))$.

(29) Let A be a universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of $\mathrm{MSAlg}(A)$. Suppose $a_2 = \mathrm{MSAlg}(a_1)$ and $b_2 = \mathrm{MSAlg}(b_1)$. Then (the sorts of $a_2) \cap ($the sorts of $b_2) = 0 \dot{\longmapsto} ($the carrier of $a_1) \cap ($the carrier of $b_1)$.

(30) Let A be a strict universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of $\mathrm{MSAlg}(A)$. If $a_2 = \mathrm{MSAlg}(a_1)$ and $b_2 = \mathrm{MSAlg}(b_1)$, then $\mathrm{MSAlg}(a_1 \sqcup b_1) = a_2 \sqcup b_2$.

Let A be a universal algebra with constants. One can verify that $\mathrm{MSSign}(A)$ is non void, strict, segmental, and trivial and has constant operations.

The following proposition is true

(31) Let A be a universal algebra with constants, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of $\mathrm{MSAlg}(A)$. If $a_2 = \mathrm{MSAlg}(a_1)$ and $b_2 = \mathrm{MSAlg}(b_1)$, then $\mathrm{MSAlg}(a_1 \cap b_1) = a_2 \cap b_2$.

Let A be a quasi total universal algebra structure. Note that the universal algebra structure of A is quasi total.

Let A be a partial universal algebra structure. Note that the universal algebra structure of A is partial.

Let A be a non-empty universal algebra structure. Observe that the universal algebra structure of A is non-empty.

Let A be a universal algebra with constants. Note that the universal algebra structure of A has constants.

We now state the proposition

(32) Let A be a universal algebra with constants. Then the lattice of subalgebras of the universal algebra structure of A and the lattice of subalgebras of MSAlg(the universal algebra structure of A) are isomorphic.

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