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# The One-Dimensional Lebesgue Measure

Józef Białas
Łódź University
Łódź

**Summary.** The paper is the crowning of a series of articles written in the Mizar language, being a formalization of notions needed for the description of the one-dimensional Lebesgue measure. The formalization of the notion as classical as the Lebesgue measure determines the powers of the PC Mizar system as a tool for the strict, precise notation and verification of the correctness of deductive theories. Following the successive articles [2], [3], [4], [8] constructed so that the final one should include the definition and the basic properties of the Lebesgue measure, we observe one of the paths relatively simple in the sense of the definition, enabling us the formal introduction of this notion. This way, although toilsome, since such is the nature of formal theories, is greatly instructive. It brings home the proper succession of the introduction of the definitions of intermediate notions and points out to those elements of the theory which determine the essence of the complexity of the notion being introduced.

The paper includes the definition of the σ-field of Lebesgue measurable sets, the definition of the Lebesgue measure and the basic set of the theorems describing its properties.

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The articles [12], [11], [15], [13], [14], [16], [9], [2], [3], [4], [5], [6], [7], [10], and [1] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For every function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number $n$ holds $F(n) = 0_{\overline{\mathbb{R}}}$ holds $\sum F = 0_{\overline{\mathbb{R}}}$.

(2) For every function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F$ is non-negative and for every natural number $n$ holds $F(n) \leq (\mathrm{Ser}\, F)(n)$.

(3) Let $F$, $G$, $H$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $G$ is non-negative and $H$ is non-negative. Suppose that for every natural number $n$ holds $F(n) = G(n) + H(n)$. Let $n$ be a natural number. Then $(\mathrm{Ser}\, F)(n) = (\mathrm{Ser}\, G)(n) + (\mathrm{Ser}\, H)(n)$.

(4) Let $F$, $G$, $H$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose that for every natural number $n$ holds $F(n) = G(n) + H(n)$. If $G$ is non-negative and $H$ is non-negative, then $\sum F \leq \sum G + \sum H$.

(6)[1] Let $F$, $G$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative and for every natural number $n$ holds $F(n) \leq G(n)$. Let $n$ be a natural number. Then $(\mathrm{Ser}\, F)(n) \leq \sum G$.

(7) For every function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F$ is non-negative and for every natural number $n$ holds $(\mathrm{Ser}\, F)(n) \leq \sum F$.

[1] The proposition (5) has been removed.

Let $S$ be a non empty subset of $\mathbb{N}$, let $H$ be a function from $S$ into $\mathbb{N}$, and let $n$ be an element of $S$. Then $H(n)$ is a natural number.

Let $S$ be a non empty set and let $H$ be a function from $S$ into $\overline{\mathbb{R}}$. The functor $\mathrm{On}H$ yielding a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by:

(Def. 1) For every element $n$ of $\mathbb{N}$ holds if $n \in S$, then $(\mathrm{On}H)(n) = H(n)$ and if $n \notin S$, then $(\mathrm{On}H)(n) = 0_{\overline{\mathbb{R}}}$.

One can prove the following propositions:

(8) For every non empty set $X$ and for every function $G$ from $X$ into $\overline{\mathbb{R}}$ such that $G$ is non-negative holds $\mathrm{On}G$ is non-negative.

(9) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. Let $n$, $k$ be natural numbers. If $n \leq k$, then $(\mathrm{Ser}F)(n) \leq (\mathrm{Ser}F)(k)$.

(10) Let $k$ be a natural number and $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose that for every natural number $n$ such that $n \neq k$ holds $F(n) = 0_{\overline{\mathbb{R}}}$. Then

(i) for every natural number $n$ such that $n < k$ holds $(\mathrm{Ser}F)(n) = 0_{\overline{\mathbb{R}}}$, and

(ii) for every natural number $n$ such that $k \leq n$ holds $(\mathrm{Ser}F)(n) = F(k)$.

(11) Let $G$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $G$ is non-negative. Let $S$ be a non empty subset of $\mathbb{N}$ and $H$ be a function from $S$ into $\mathbb{N}$. If $H$ is one-to-one, then $\sum \mathrm{On}(G \cdot H) \leq \sum G$.

(12) Let $F$, $G$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative and $G$ is non-negative. Let $S$ be a non empty subset of $\mathbb{N}$ and $H$ be a function from $S$ into $\mathbb{N}$. Suppose $H$ is one-to-one. Suppose that for every natural number $k$ holds if $k \in S$, then $F(k) = (G \cdot H)(k)$ and if $k \notin S$, then $F(k) = 0_{\overline{\mathbb{R}}}$. Then $\sum F \leq \sum G$.

Let $A$ be a subset of $\mathbb{R}$. A function from $\mathbb{N}$ into $2^{\mathbb{R}}$ is said to be an interval covering of $A$ if:

(Def. 2) $A \subseteq \bigcup \mathrm{rng}\,\mathrm{it}$ and for every natural number $n$ holds $\mathrm{it}(n)$ is an interval.

Let $A$ be a subset of $\mathbb{R}$, let $F$ be an interval covering of $A$, and let $n$ be a natural number. Then $F(n)$ is an interval.

Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. A function from $\mathbb{N}$ into $(2^{\mathbb{R}})^{\mathbb{N}}$ is said to be an interval covering of $F$ if:

(Def. 3) For every natural number $n$ holds $\mathrm{it}(n)$ is an interval covering of $F(n)$.

Let $A$ be a subset of $\mathbb{R}$ and let $F$ be an interval covering of $A$. The functor $(F)\,\mathrm{vol}$ yielding a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is defined as follows:

(Def. 4) For every natural number $n$ holds $(F)\,\mathrm{vol}(n) = \mathrm{vol}(F(n))$.

Next we state the proposition

(13) For every subset $A$ of $\mathbb{R}$ and for every interval covering $F$ of $A$ holds $(F)\,\mathrm{vol}$ is non-negative.

Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let $H$ be an interval covering of $F$, and let $n$ be a natural number. Then $H(n)$ is an interval covering of $F(n)$.

Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let $G$ be an interval covering of $F$. The functor $(G)\,\mathrm{vol}$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}^{\mathbb{N}}$ and is defined as follows:

(Def. 5) For every natural number $n$ holds $(G)\,\mathrm{vol}(n) = (G(n))\,\mathrm{vol}$.

Let $A$ be a subset of $\mathbb{R}$ and let $F$ be an interval covering of $A$. The functor $\mathrm{vol}(F)$ yielding an extended real number is defined as follows:

(Def. 6) $\mathrm{vol}(F) = \sum((F)\,\mathrm{vol})$.

Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let $G$ be an interval covering of $F$. The functor $\mathrm{vol}(G)$ yielding a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) For every natural number $n$ holds $(\mathrm{vol}(G))(n) = \mathrm{vol}(G(n))$.

One can prove the following proposition

(14) Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, $G$ be an interval covering of $F$, and $n$ be a natural number. Then $0_{\overline{\mathbb{R}}} \leq (\mathrm{vol}(G))(n)$.

Let $A$ be a subset of $\mathbb{R}$. The functor $\mathrm{Svc}(A)$ yielding a subset of $\overline{\mathbb{R}}$ is defined by:

(Def. 8) For every extended real number $x$ holds $x \in \mathrm{Svc}(A)$ iff there exists an interval covering $F$ of $A$ such that $x = \mathrm{vol}(F)$.

Let $A$ be a subset of $\mathbb{R}$. One can check that $\mathrm{Svc}(A)$ is non empty.
Let $A$ be an element of $2^{\mathbb{R}}$. The functor $\mathbb{C}^A$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 9) $\mathbb{C}^A = \inf \mathrm{Svc}(A)$.

The function OSMeas from $2^{\mathbb{R}}$ into $\overline{\mathbb{R}}$ is defined as follows:

(Def. 10) For every subset $A$ of $\mathbb{R}$ holds $(\mathrm{OSMeas})(A) = \inf \mathrm{Svc}(A)$.

Let $H$ be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Then $\mathrm{pr1}(H)$ is a function from $\mathbb{N}$ into $\mathbb{N}$ and it can be characterized by the condition:

(Def. 11) For every element $n$ of $\mathbb{N}$ there exists an element $s$ of $\mathbb{N}$ such that $H(n) = \langle \mathrm{pr1}(H)(n), s \rangle$.

Let $H$ be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Then $\mathrm{pr2}(H)$ is a function from $\mathbb{N}$ into $\mathbb{N}$ and it can be characterized by the condition:

(Def. 12) For every element $n$ of $\mathbb{N}$ holds $H(n) = \langle \mathrm{pr1}(H)(n), \mathrm{pr2}(H)(n) \rangle$.

Let $F$ be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let $G$ be an interval covering of $F$, and let $H$ be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Let us assume that $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. The functor $\mathrm{On}(G,H)$ yields an interval covering of $\bigcup \mathrm{rng}\, F$ and is defined by:

(Def. 13) For every element $n$ of $\mathbb{N}$ holds $(\mathrm{On}(G,H))(n) = G(\mathrm{pr1}(H)(n))(\mathrm{pr2}(H)(n))$.

The following propositions are true:

(15) Let $H$ be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Suppose $H$ is one-to-one and $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. Let $k$ be a natural number. Then there exists a natural number $m$ such that for every function $F$ from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering $G$ of $F$ holds $(\mathrm{Ser}((\mathrm{On}(G,H))\, \mathrm{vol}))(k) \leq (\mathrm{Ser}\, \mathrm{vol}(G))(m)$.

(16) For every function $F$ from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering $G$ of $F$ holds $\inf \mathrm{Svc}(\bigcup \mathrm{rng}\, F) \leq \sum \mathrm{vol}(G)$.

(17) OSMeas is a Caratheodor's measure on $\mathbb{R}$.

OSMeas is a Caratheodor's measure on $\mathbb{R}$.
The functor $L_\mu$-σFIELD is a σ-field of subsets of $\mathbb{R}$ and is defined by:

(Def. 14) $L_\mu$-σFIELD = σ-Field(OSMeas).

The σ-measure $L_\mu$ on $L_\mu$-σFIELD is defined by:

(Def. 15) $L_\mu$ = σ-Meas(OSMeas).

Next we state two propositions:

(18) $L_\mu$ is complete on $L_\mu$-σFIELD.

(21)[2] For every set $A$ such that $A \in L_\mu$-σFIELD holds $\mathbb{R} \setminus A \in L_\mu$-σFIELD.

[2] The propositions (19) and (20) have been removed.

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