JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Properties of Caratheodor's Measure

Józef Białas
University of Łódź

Summary. The paper contains definitions and basic properties of Caratheodor's measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [10]. The article includes the text being a continuation of the paper [5]. Caratheodor's theorem and some theorems concerning basic properties of Caratheodor's measure are proved. The work is the sixth part of the series of articles concerning the Lebesgue measure theory.

MML Identifier: `MEASURE4`.

WWW: `http://mizar.org/JFM/Vol4/measure4.html`

The articles [11], [8], [13], [12], [14], [6], [7], [1], [9], [2], [3], [4], and [5] provide the notation and terminology for this paper.

We adopt the following convention: x, y, z denote extended real numbers, A, B, X denote sets, and S denotes a σ-field of subsets of X.

We now state several propositions:

(1) If $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$, then $(x+y)+z = x+(y+z)$.

(2) If $x \neq -\infty$ and $x \neq +\infty$, then $y+x \leq z$ iff $y \leq z-x$.

(3) If $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$, then $x+y = y+x$.

(4) Let S be a non empty family of subsets of X, F, G be functions from $\mathbb{N}$ into S, and A be an element of S. If for every element n of $\mathbb{N}$ holds $G(n) = A \cap F(n)$, then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(5) Let S be a non empty family of subsets of X and F, G be functions from $\mathbb{N}$ into S. Suppose $G(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(n+1) \cup G(n)$. Let H be a function from $\mathbb{N}$ into S. Suppose $H(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $H(n+1) = F(n+1) \setminus G(n)$. Then $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} H$.

(6) 2^X is a σ-field of subsets of X.

Let X be a set and let F be a function from $\mathbb{N}$ into 2^X. Then $\operatorname{rng} F$ is a family of subsets of X.

Let X be a set and let A be a family of subsets of X. Then $\bigcup A$ is an element of 2^X.

Let Y be a set, let X, Z be non empty sets, let F be a function from Y into X, and let M be a function from X into Z. Then $M \cdot F$ is a function from Y into Z.

The following three propositions are true:

(7) Let a, b be extended real numbers. Then there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds

(i) if $A = \emptyset$, then $M(A) = a$, and

(ii) if $A \neq \emptyset$, then $M(A) = b$.

(8) There exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(11)[1] There exists a function M from 2^X into $\overline{\mathbb{R}}$ such that

(i) M is non-negative,

(ii) $M(\emptyset) = 0_{\overline{\mathbb{R}}}$, and

(iii) for all elements A, B of 2^X such that $A \subseteq B$ holds $M(A) \leq M(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$.

Let X be a set. A function from 2^X into $\overline{\mathbb{R}}$ is said to be a Caratheodor's measure on X if it satisfies the conditions (Def. 2).

(Def. 2)[2](i) It is non-negative,

(ii) $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$, and

(iii) for all elements A, B of 2^X such that $A \subseteq B$ holds $\mathrm{it}(A) \leq \mathrm{it}(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $\mathrm{it}(\bigcup \operatorname{rng} F) \leq \sum(\mathrm{it} \cdot F)$.

In the sequel C denotes a Caratheodor's measure on X.

Let X be a set and let C be a Caratheodor's measure on X. The functor σ-Field(C) yields a non empty family of subsets of X and is defined by the condition (Def. 3).

(Def. 3) Let A be an element of 2^X. Then $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) \leq C(W \cup Z)$.

One can prove the following propositions:

(12) For all elements W, Z of 2^X holds $C(W \cup Z) \leq C(W) + C(Z)$.

(13) For all elements W, Z of 2^X holds $C(Z) + C(W) = C(W) + C(Z)$.

(14) Let A be an element of 2^X. Then $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) = C(W \cup Z)$.

(15) For all elements W, Z of 2^X such that $W \in \sigma$-Field(C) and $Z \in \sigma$-Field(C) and Z misses W holds $C(W \cup Z) = C(W) + C(Z)$.

(16) If $A \in \sigma$-Field(C), then $X \setminus A \in \sigma$-Field(C).

(17) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \cup B \in \sigma$-Field(C).

(18) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \cap B \in \sigma$-Field(C).

(19) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \setminus B \in \sigma$-Field(C).

(20) Let N be a function from $\mathbb{N}$ into S and A be an element of S. Then there exists a function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) = A \cap N(n)$.

(21) σ-Field(C) is a σ-field of subsets of X.

Let X be a set and let C be a Caratheodor's measure on X. One can check that σ-Field(C) is σ-field of subsets-like, closed for complement operator, and non empty.

Let X be a set, let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S.

Let X be a set and let C be a Caratheodor's measure on X. The functor σ-Meas(C) yielding a function from σ-Field(C) into $\overline{\mathbb{R}}$ is defined as follows:

(Def. 4) For every element A of 2^X such that $A \in \sigma$-Field(C) holds $(\sigma\text{-Meas}(C))(A) = C(A)$.

[1] The propositions (9) and (10) have been removed.

[2] The definition (Def. 1) has been removed.

We now state the proposition

(22) σ-Meas(C) is a measure on σ-Field(C).

Let X be a set, let C be a Caratheodor's measure on X, and let A be an element of σ-Field(C). Then $C(A)$ is an extended real number.

One can prove the following proposition

(23) σ-Meas(C) is a σ-measure on σ-Field(C).

Let X be a set and let C be a Caratheodor's measure on X. Then σ-Meas(C) is a σ-measure on σ-Field(C).

One can prove the following two propositions:

(24) For every element A of 2^X such that $C(A) = 0_{\overline{\mathbb{R}}}$ holds $A \in$ σ-Field(C).

(25) σ-Meas(C) is complete on σ-Field(C).

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_1.html.

[3] Józef Białas. Series of positive real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_2.html.

[4] Józef Białas. The σ-additive measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/measure1.html.

[5] Józef Białas. Completeness of the σ-additive measure. Measure theory. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/measure3.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[10] R. Sikorski. *Rachunek różniczkowy i całkowy - funkcje wielu zmiennych*. Biblioteka Matematyczna. PWN - Warszawa, 1968.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received June 25, 1992

Published January 2, 2004