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Completeness of the σ-Additive Measure. Measure Theory

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Summary. Definitions and basic properties of a σ-additive, non-negative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [10]. The article includes the text being a continuation of the paper [5]. Some theorems concerning basic properties of a σ-additive measure and completeness of the measure are proved.

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The articles [11], [8], [13], [12], [14], [6], [7], [1], [9], [2], [3], [4], and [5] provide the notation and terminology for this paper.

In this paper X denotes a set.

One can prove the following four propositions:

(1) For every extended real number x such that $-\infty < x$ and $x < +\infty$ holds x is a real number.

(2) For every extended real number x such that $x \neq -\infty$ and $x \neq +\infty$ holds x is a real number.

(3) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $(\mathrm{Ser}\,F_1)(n) \leq (\mathrm{Ser}\,F_2)(n)$ holds $\sum F_1 \leq \sum F_2$.

(4) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $(\mathrm{Ser}\,F_1)(n) = (\mathrm{Ser}\,F_2)(n)$ holds $\sum F_1 = \sum F_2$.

Let X be a set and let S be a σ-field of subsets of X. We introduce subfamily of S as a synonym of family of measurable sets of S.

Let X be a set, let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S. Then $\mathrm{rng}\,F$ is a family of measurable sets of S.

Next we state a number of propositions:

(5) Let S be a σ-field of subsets of X, M be a σ-measure on S, F be a function from $\mathbb{N}$ into S, and A be an element of S. If $\bigcap \mathrm{rng}\,F \subseteq A$ and for every element n of $\mathbb{N}$ holds $A \subseteq F(n)$, then $M(A) = M(\bigcap \mathrm{rng}\,F)$.

(6) Let S be a σ-field of subsets of X and G, F be functions from $\mathbb{N}$ into S. Suppose $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\bigcup \mathrm{rng}\,G = F(0) \setminus \bigcap \mathrm{rng}\,F$.

(7) Let S be a σ-field of subsets of X and G, F be functions from $\mathbb{N}$ into S. Suppose $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\bigcap \mathrm{rng}\,F = F(0) \setminus \bigcup \mathrm{rng}\,G$.

(8) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - M(\bigcup \operatorname{rng} G)$.

(9) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcup \operatorname{rng} G) = M(F(0)) - M(\bigcap \operatorname{rng} F)$.

(10) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(11) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G)$ is a real number and $M(F(0))$ is a real number and $\inf \operatorname{rng}(M \cdot F)$ is a real number.

(12) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G) = M(F(0)) - \inf \operatorname{rng}(M \cdot F)$.

(13) Let S be a σ-field of subsets of X, M be a σ-measure on S, and G, F be functions from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\inf \operatorname{rng}(M \cdot F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(14) Let S be a σ-field of subsets of X, M be a σ-measure on S, and F be a function from $\mathbb{N}$ into S. Suppose for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$ and $M(F(0)) < +\infty$. Then $M(\bigcap \operatorname{rng} F) = \inf \operatorname{rng}(M \cdot F)$.

(15) Let S be a σ-field of subsets of X, M be a measure on S, T be a family of measurable sets of S, and F be a sequence of separated subsets of S. If $T = \operatorname{rng} F$, then $\sum(M \cdot F) \leq M(\bigcup T)$.

(16) Let S be a σ-field of subsets of X, M be a measure on S, and F be a sequence of separated subsets of S. Then $\sum(M \cdot F) \leq M(\bigcup \operatorname{rng} F)$.

(17) Let S be a σ-field of subsets of X and M be a measure on S. Suppose that for every sequence F of separated subsets of S holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$. Then M is a σ-measure on S.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. We say that M is complete on S if and only if:

(Def. 2)[1] For every subset A of X and for every set B such that $B \in S$ holds if $A \subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$, then $A \in S$.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. A subset of X is called a set with measure zero w.r.t. M if:

(Def. 3) There exists a set B such that $B \in S$ and it $\subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(S, M)$ yielding a non empty family of subsets of X is defined by the condition (Def. 4).

(Def. 4) Let A be a set. Then $A \in \mathrm{COM}(S, M)$ if and only if there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$.

Let X be a set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let A be an element of $\mathrm{COM}(S, M)$. The functor MeasPart$A$ yielding a non empty family of subsets of X is defined as follows:

(Def. 5) For every set B holds $B \in$ MeasPart$A$ iff $B \in S$ and $B \subseteq A$ and $A \setminus B$ is a set with measure zero w.r.t. M.

[1] The definition (Def. 1) has been removed.

Next we state four propositions:

(18) Let S be a σ-field of subsets of X, M be a σ-measure on S, and F be a function from $\mathbb{N}$ into $\mathrm{COM}(S,M)$. Then there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $G(n) \in \mathrm{MeasPart}F(n)$.

(19) Let S be a σ-field of subsets of X, M be a σ-measure on S, F be a function from $\mathbb{N}$ into $\mathrm{COM}(S,M)$, and G be a function from $\mathbb{N}$ into S. Then there exists a function H from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$ holds $H(n) = F(n) \setminus G(n)$.

(20) Let S be a σ-field of subsets of X, M be a σ-measure on S, and F be a function from $\mathbb{N}$ into 2^X. Suppose that for every element n of $\mathbb{N}$ holds $F(n)$ is a set with measure zero w.r.t. M. Then there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds

$F(n) \subseteq G(n)$ and $M(G(n)) = 0_{\overline{\mathbb{R}}}$.

(21) Let S be a σ-field of subsets of X, M be a σ-measure on S, and D be a non empty family of subsets of X. Suppose that for every set A holds $A \in D$ iff there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$. Then D is a σ-field of subsets of X.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can verify that $\mathrm{COM}(S,M)$ is σ-field of subsets-like, closed for complement operator, and non empty.

We now state the proposition

(22) Let S be a σ-field of subsets of X, M be a σ-measure on S, and B_1, B_2 be sets. Suppose $B_1 \in S$ and $B_2 \in S$. Let C_1, C_2 be sets with measure zero w.r.t. M. If $B_1 \cup C_1 = B_2 \cup C_2$, then $M(B_1) = M(B_2)$.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(M)$ yielding a σ-measure on $\mathrm{COM}(S,M)$ is defined by:

(Def. 6) For every set B such that $B \in S$ and for every set C with measure zero w.r.t. M holds $(\mathrm{COM}(M))(B \cup C) = M(B)$.

We now state the proposition

(23) For every σ-field S of subsets of X and for every σ-measure M on S holds $\mathrm{COM}(M)$ is complete on $\mathrm{COM}(S,M)$.

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