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Several Properties of the σ-additive Measure

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Summary. A continuation of [4]. The paper contains the definition and basic properties of a σ-additive, nonnegative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ — by R. Sikorski [9]. Some simple theorems concerning basic properties of a σ-additive measure, measurable sets, measure zero sets are proved. The work is the fourth part of the series of articles concerning the Lebesgue measure theory.

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The articles [10], [7], [12], [11], [13], [5], [6], [1], [8], [2], [3], and [4] provide the notation and terminology for this paper.

In this paper X denotes a set.

We now state the proposition

(1) Let S be a σ-field of subsets of X, M be a σ-measure on S, and F be a function from $\mathbb{N}$ into S. Then $M \cdot F$ is non-negative.

Let X be a set and let S be a σ-field of subsets of X. A denumerable family of subsets of X is said to be a family of measurable sets of S if:

(Def. 1) It $\subseteq S$.

One can prove the following proposition

(3)[1] For every σ-field S of subsets of X and for every family T of measurable sets of S holds $\bigcap T \in S$ and $\bigcup T \in S$.

Let X be a set, let S be a σ-field of subsets of X, and let T be a family of measurable sets of S. Then $\bigcap T$ is an element of S. Then $\bigcup T$ is an element of S.

We now state a number of propositions:

(4) Let S be a σ-field of subsets of X and N be a function from $\mathbb{N}$ into S. Then there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$.

(5) Let S be a σ-field of subsets of X and N be a function from $\mathbb{N}$ into S. Then there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$.

[1] The proposition (2) has been removed.

(6) Let S be a non empty family of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$. Let r be a set and n be a natural number. Then $r \in F(n)$ if and only if there exists a natural number k such that $k \leq n$ and $r \in N(k)$.

(7) Let S be a non empty family of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$. Let n, m be natural numbers. If $n < m$, then $F(n) \subseteq F(m)$.

(8) Let S be a non empty family of subsets of X and N, G, F be functions from $\mathbb{N}$ into S. Suppose that

(i) $G(0) = N(0)$,

(ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,

(iii) $F(0) = N(0)$, and

(iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

Let n, m be natural numbers. If $n \leq m$, then $F(n) \subseteq G(m)$.

(9) Let S be a σ-field of subsets of X and N, G be functions from $\mathbb{N}$ into S. Then there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

(10) Let S be a σ-field of subsets of X and N be a function from $\mathbb{N}$ into S. Then there exists a function F from $\mathbb{N}$ into S such that $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$.

(11) Let S be a σ-field of subsets of X and N, G, F be functions from $\mathbb{N}$ into S. Suppose that

(i) $G(0) = N(0)$,

(ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,

(iii) $F(0) = N(0)$, and

(iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

Let n, m be natural numbers. If $n < m$, then $F(n)$ misses $F(m)$.

(13)[2] Let S be a σ-field of subsets of X, M be a σ-measure on S, T be a family of measurable sets of S, and F be a function from $\mathbb{N}$ into S. If $T = \operatorname{rng} F$, then $M(\bigcup T) \leq \sum(M \cdot F)$.

(14) Let S be a σ-field of subsets of X and T be a family of measurable sets of S. Then there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$.

(15) Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$. Let n be an element of $\mathbb{N}$. Then $F(n) \subseteq F(n+1)$.

(16) Let S be a σ-field of subsets of X, M be a σ-measure on S, and T be a family of measurable sets of S. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M. Then $\bigcup T$ is a set of measure zero w.r.t. M.

(17) Let S be a σ-field of subsets of X, M be a σ-measure on S, and T be a family of measurable sets of S. Given a set A such that $A \in T$ and A is a set of measure zero w.r.t. M. Then $\bigcap T$ is a set of measure zero w.r.t. M.

(18) Let S be a σ-field of subsets of X, M be a σ-measure on S, and T be a family of measurable sets of S. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M. Then $\bigcap T$ is a set of measure zero w.r.t. M.

Let X be a set, let S be a σ-field of subsets of X, and let I_1 be a family of measurable sets of S. We say that I_1 is non-decreasing if and only if:

[2] The proposition (12) has been removed.

(Def. 2) There exists a function F from $\mathbb{N}$ into S such that $I_1 = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

Let X be a set and let S be a σ-field of subsets of X. Note that there exists a family of measurable sets of S which is non-decreasing.

Let X be a set, let S be a σ-field of subsets of X, and let I_1 be a family of measurable sets of S. We say that I_1 is non-increasing if and only if:

(Def. 3) There exists a function F from $\mathbb{N}$ into S such that $I_1 = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$.

Let X be a set and let S be a σ-field of subsets of X. Note that there exists a family of measurable sets of S which is non-increasing.

One can prove the following propositions:

(21)[3] Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$. Then $\operatorname{rng} F$ is a non-decreasing family of measurable sets of S.

(22) Let S be a non empty family of subsets of X and N be a function from $\mathbb{N}$ into S. Suppose that for every element n of $\mathbb{N}$ holds $N(n) \subseteq N(n+1)$. Let m, n be natural numbers. If $n < m$, then $N(n) \subseteq N(m)$.

(23) Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Let n, m be natural numbers. If $n < m$, then $F(n)$ misses $F(m)$.

(24) Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} N$.

(25) Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then F is a sequence of separated subsets of S.

(26) Let S be a σ-field of subsets of X and N, F be functions from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then $N(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $N(n+1) = F(n+1) \cup N(n)$.

(27) Let S be a σ-field of subsets of X, M be a σ-measure on S, and F be a function from $\mathbb{N}$ into S. If for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$, then $M(\bigcup \operatorname{rng} F) = \sup \operatorname{rng}(M \cdot F)$.

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