JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

N-Tuples and Cartesian Products for $n = 5$[1]

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Summary. This article defines ordered n-tuples, projections and Cartesian products for $n = 5$. We prove many theorems concerning the basic properties of the n-tuples and Cartesian products that may be utilized in several further, more challenging applications. A few of these theorems are a strightforward consequence of the regularity axiom. The article originated as an upgrade of the article [3].

MML Identifier: MCART_2.

WWW: http://mizar.org/JFM/Vol2/mcart_2.html

The articles [2], [1], [4], and [3] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x, x_1, x_2, x_3, x_4, x_5, y, y_1, y_2, y_3, y_4, y_5, X, X_1, X_2, X_3, X_4, X_5, Y, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Z are sets, x_6 is an element of X_1, x_7 is an element of X_2, x_8 is an element of X_3, x_9 is an element of X_4, and x_{10} is an element of X_5.

Next we state two propositions:

(1) Suppose $X \neq \emptyset$. Then there exists Y such that $Y \in X$ and for all Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y$ holds Y_1 misses X.

(2) Suppose $X \neq \emptyset$. Then there exists Y such that $Y \in X$ and for all Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7 such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y_7$ and $Y_7 \in Y$ holds Y_1 misses X.

Let us consider x_1, x_2, x_3, x_4, x_5. The functor $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is defined as follows:

(Def. 1) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2, x_3, x_4 \rangle, x_5 \rangle$.

We now state several propositions:

(3) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle \langle \langle x_1, x_2 \rangle, x_3 \rangle, x_4 \rangle, x_5 \rangle$.

(5)[1] $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2, x_3 \rangle, x_4, x_5 \rangle$.

(6) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2 \rangle, x_3, x_4, x_5 \rangle$.

(7) If $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle y_1, y_2, y_3, y_4, y_5 \rangle$, then $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$ and $x_5 = y_5$.

(8) If $X \neq \emptyset$, then there exists x such that $x \in X$ and it is not true that there exist x_1, x_2, x_3, x_4, x_5 such that $x_1 \in X$ or $x_2 \in X$ but $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

[1]Supported by RPBP.III-24.C6.

[1] The proposition (4) has been removed.

Let us consider X_1, X_2, X_3, X_4, X_5. The functor $[: X_1, X_2, X_3, X_4, X_5 :]$ yields a set and is defined as follows:

(Def. 2) $[: X_1, X_2, X_3, X_4, X_5 :] = [: [: X_1, X_2, X_3, X_4 :], X_5 :]$.

Next we state several propositions:

(9) $[: X_1, X_2, X_3, X_4, X_5 :] = [: [: [: [: X_1, X_2 :], X_3 :], X_4 :], X_5 :]$.

(11)[2] $[: X_1, X_2, X_3, X_4, X_5 :] = [: [: X_1, X_2, X_3 :], X_4, X_5 :]$.

(12) $[: X_1, X_2, X_3, X_4, X_5 :] = [: [: X_1, X_2 :], X_3, X_4, X_5 :]$.

(13) $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ iff $[: X_1, X_2, X_3, X_4, X_5 :] \neq \emptyset$.

(14) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$, then if $[: X_1, X_2, X_3, X_4, X_5 :] = [: Y_1, Y_2, Y_3, Y_4, Y_5 :]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$.

(15) If $[: X_1, X_2, X_3, X_4, X_5 :] \neq \emptyset$ and $[: X_1, X_2, X_3, X_4, X_5 :] = [: Y_1, Y_2, Y_3, Y_4, Y_5 :]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$.

(16) If $[: X, X, X, X, X :] = [: Y, Y, Y, Y, Y :]$, then $X = Y$.

(17) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$, then for every element x of $[: X_1, X_2, X_3, X_4, X_5 :]$ there exist x_6, x_7, x_8, x_9, x_{10} such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$.

Let us consider X_1, X_2, X_3, X_4, X_5. Let us assume that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5 :]$. The functor $x_{\mathbf{1}}$ yields an element of X_1 and is defined by:

(Def. 3) If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{1}} = x_1$.

The functor $x_{\mathbf{2}}$ yields an element of X_2 and is defined by:

(Def. 4) If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{2}} = x_2$.

The functor $x_{\mathbf{3}}$ yielding an element of X_3 is defined as follows:

(Def. 5) If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{3}} = x_3$.

The functor $x_{\mathbf{4}}$ yielding an element of X_4 is defined as follows:

(Def. 6) If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{4}} = x_4$.

The functor $x_{\mathbf{5}}$ yields an element of X_5 and is defined by:

(Def. 7) If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{5}} = x_5$.

The following propositions are true:

(18) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5 :]$ and given x_1, x_2, x_3, x_4, x_5. If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{1}} = x_1$ and $x_{\mathbf{2}} = x_2$ and $x_{\mathbf{3}} = x_3$ and $x_{\mathbf{4}} = x_4$ and $x_{\mathbf{5}} = x_5$.

(19) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$, then for every element x of $[: X_1, X_2, X_3, X_4, X_5 :]$ holds $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}, x_{\mathbf{4}}, x_{\mathbf{5}} \rangle$.

(20) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5 :]$. Then $x_{\mathbf{1}} = ((((x \text{ **qua** set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}}$ and $x_{\mathbf{2}} = ((((x \text{ **qua** set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{2}}$ and $x_{\mathbf{3}} = (((x \text{ **qua** set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{2}}$ and $x_{\mathbf{4}} = ((x \text{ **qua** set})_{\mathbf{1}})_{\mathbf{2}}$ and $x_{\mathbf{5}} = (x \text{ **qua** set})_{\mathbf{2}}$.

(21) If $X_1 \subseteq [: X_1, X_2, X_3, X_4, X_5 :]$ or $X_1 \subseteq [: X_2, X_3, X_4, X_5, X_1 :]$ or $X_1 \subseteq [: X_3, X_4, X_5, X_1, X_2 :]$ or $X_1 \subseteq [: X_4, X_5, X_1, X_2, X_3 :]$ or $X_1 \subseteq [: X_5, X_1, X_2, X_3, X_4 :]$, then $X_1 = \emptyset$.

[2] The proposition (10) has been removed.

(22) If $[: X_1, X_2, X_3, X_4, X_5 :]$ meets $[: Y_1, Y_2, Y_3, Y_4, Y_5 :]$, then X_1 meets Y_1 and X_2 meets Y_2 and X_3 meets Y_3 and X_4 meets Y_4 and X_5 meets Y_5.

(23) $[: \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\} :] = \{\langle x_1, x_2, x_3, x_4, x_5 \rangle\}$.

For simplicity, we use the following convention: A_1 is a subset of X_1, A_2 is a subset of X_2, A_3 is a subset of X_3, A_4 is a subset of X_4, A_5 is a subset of X_5, and x is an element of $[: X_1, X_2, X_3, X_4, X_5 :]$.

One can prove the following propositions:

(24) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let given x_1, x_2, x_3, x_4, x_5. If $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $x_{\mathbf{1}} = x_1$ and $x_{\mathbf{2}} = x_2$ and $x_{\mathbf{3}} = x_3$ and $x_{\mathbf{4}} = x_4$ and $x_{\mathbf{5}} = x_5$.

(25) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_1 = x_6$, then $y_1 = x_{\mathbf{1}}$.

(26) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_2 = x_7$, then $y_2 = x_{\mathbf{2}}$.

(27) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_3 = x_8$, then $y_3 = x_{\mathbf{3}}$.

(28) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_4 = x_9$, then $y_4 = x_{\mathbf{4}}$.

(29) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_5 = x_{10}$, then $y_5 = x_{\mathbf{5}}$.

(30) If $y \in [: X_1, X_2, X_3, X_4, X_5 :]$, then there exist x_1, x_2, x_3, x_4, x_5 such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $y = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

(31) $\langle x_1, x_2, x_3, x_4, x_5 \rangle \in [: X_1, X_2, X_3, X_4, X_5 :]$ iff $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$.

(32) Suppose that for every y holds $y \in Z$ iff there exist x_1, x_2, x_3, x_4, x_5 such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $y = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. Then $Z = [: X_1, X_2, X_3, X_4, X_5 :]$.

(33) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ and $Y_3 \neq \emptyset$ and $Y_4 \neq \emptyset$ and $Y_5 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5 :]$ and y be an element of $[: Y_1, Y_2, Y_3, Y_4, Y_5 :]$. If $x = y$, then $x_{\mathbf{1}} = y_{\mathbf{1}}$ and $x_{\mathbf{2}} = y_{\mathbf{2}}$ and $x_{\mathbf{3}} = y_{\mathbf{3}}$ and $x_{\mathbf{4}} = y_{\mathbf{4}}$ and $x_{\mathbf{5}} = y_{\mathbf{5}}$.

(34) For every element x of $[: X_1, X_2, X_3, X_4, X_5 :]$ such that $x \in [: A_1, A_2, A_3, A_4, A_5 :]$ holds $x_{\mathbf{1}} \in A_1$ and $x_{\mathbf{2}} \in A_2$ and $x_{\mathbf{3}} \in A_3$ and $x_{\mathbf{4}} \in A_4$ and $x_{\mathbf{5}} \in A_5$.

(35) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ and $X_3 \subseteq Y_3$ and $X_4 \subseteq Y_4$ and $X_5 \subseteq Y_5$, then $[: X_1, X_2, X_3, X_4, X_5 :] \subseteq [: Y_1, Y_2, Y_3, Y_4, Y_5 :]$.

Let us consider $X_1, X_2, X_3, X_4, X_5, A_1, A_2, A_3, A_4, A_5$. Then $[: A_1, A_2, A_3, A_4, A_5 :]$ is a subset of $[: X_1, X_2, X_3, X_4, X_5 :]$.

One can prove the following three propositions:

(36) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Let x_{11} be an element of $[: X_1, X_2 :]$. Then there exists an element x_6 of X_1 and there exists an element x_7 of X_2 such that $x_{11} = \langle x_6, x_7 \rangle$.

(37) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$, then for every element x_{11} of $[: X_1, X_2, X_3 :]$ there exist x_6, x_7, x_8 such that $x_{11} = \langle x_6, x_7, x_8 \rangle$.

(38) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$, then for every element x_{11} of $[: X_1, X_2, X_3, X_4 :]$ there exist x_6, x_7, x_8, x_9 such that $x_{11} = \langle x_6, x_7, x_8, x_9 \rangle$.

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Received October 13, 1990

Published January 2, 2004