JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Matrices. Abelian Group of Matrices

Katarzyna Jankowska
Warsaw University
Białystok

Summary. The basic conceptions of matrix algebra are introduced. The matrix is introduced as the finite sequence of sequences with the same length, i.e. as a sequence of lines. There are considered matrices over a field, and the fact that these matrices with addition form an Abelian group is proved.

MML Identifier: `MATRIX_1`.

WWW: `http://mizar.org/JFM/Vol3/matrix_1.html`

The articles [11], [6], [13], [14], [4], [5], [2], [10], [8], [3], [7], [12], [9], and [1] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x is a set, i, j, n, m are natural numbers, D is a non empty set, K is a non empty double loop structure, s is a finite sequence, a, a_1, a_2, b_1, b_2, d are elements of D, p, p_1, p_2 are finite sequences of elements of D, and F is an add-associative right zeroed right complementable Abelian non empty double loop structure.

Let f be a finite sequence. We say that f is tabular if and only if:

(Def. 1) There exists a natural number n such that for every x such that $x \in \operatorname{rng} f$ there exists s such that $s = x$ and $\operatorname{len} s = n$.

Let us observe that there exists a finite sequence which is tabular.

One can prove the following propositions:

(1) $\langle\langle d\rangle\rangle$ is tabular.

(2) $m \longmapsto (n \longmapsto x)$ is tabular.

(3) For every s holds $\langle s\rangle$ is tabular.

(4) For all finite sequences s_1, s_2 such that $\operatorname{len} s_1 = n$ and $\operatorname{len} s_2 = n$ holds $\langle s_1, s_2\rangle$ is tabular.

(5) $\emptyset$ is tabular.

(6) $\langle\emptyset,\emptyset\rangle$ is tabular.

(7) $\langle\langle a_1\rangle,\langle a_2\rangle\rangle$ is tabular.

(8) $\langle\langle a_1,a_2\rangle,\langle b_1,b_2\rangle\rangle$ is tabular.

Let f be a binary relation. We say that f is empty yielding if and only if:

(Def. 2) For every set s such that $s \in \operatorname{rng} f$ holds $\overline{\overline{s}} = 0$.

Let D be a set. One can verify that there exists a finite sequence of elements of D^* which is tabular.

Let D be a set. A matrix over D is a tabular finite sequence of elements of D^*.

Let D be a non empty set. Observe that there exists a matrix over D which is non empty yielding.

We now state the proposition

(9) s is a matrix over D iff there exists n such that for every x such that $x \in \operatorname{rng} s$ there exists p such that $x = p$ and $\operatorname{len} p = n$.

Let us consider D, m, n. A matrix over D is called a matrix over D of dimension $m \times n$ if:

(Def. 3) $\operatorname{len} \mathrm{it} = m$ and for every p such that $p \in \operatorname{rng} \mathrm{it}$ it holds $\operatorname{len} p = n$.

Let us consider D, n. A matrix over D of dimension n is a matrix over D of dimension $n \times n$.

Let K be a non empty 1-sorted structure. A matrix over K is a matrix over the carrier of K. Let us consider n. A matrix over K of dimension n is a matrix over the carrier of K of dimension $n \times n$. Let us consider m. A matrix over K of dimension $n \times m$ is a matrix over the carrier of K of dimension $n \times m$.

We now state a number of propositions:

(10) $m \mapsto (n \mapsto a)$ is a matrix over D of dimension $m \times n$.

(11) For every finite sequence p of elements of D holds $\langle p \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} p$.

(12) For all p_1, p_2 such that $\operatorname{len} p_1 = n$ and $\operatorname{len} p_2 = n$ holds $\langle p_1, p_2 \rangle$ is a matrix over D of dimension $2 \times n$.

(13) $\emptyset$ is a matrix over D of dimension $0 \times m$.

(14) $\langle \emptyset \rangle$ is a matrix over D of dimension 1×0.

(15) $\langle \langle a \rangle \rangle$ is a matrix over D of dimension 1.

(16) $\langle \emptyset, \emptyset \rangle$ is a matrix over D of dimension 2×0.

(17) $\langle \langle a_1, a_2 \rangle \rangle$ is a matrix over D of dimension 1×2.

(18) $\langle \langle a_1 \rangle, \langle a_2 \rangle \rangle$ is a matrix over D of dimension 2×1.

(19) $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle$ is a matrix over D of dimension 2.

In the sequel M, M_1, M_2 denote matrices over D.

Let M be a tabular finite sequence. The functor $\operatorname{width} M$ yields a natural number and is defined by:

(Def. 4)(i) There exists s such that $s \in \operatorname{rng} M$ and $\operatorname{len} s = \operatorname{width} M$ if $\operatorname{len} M > 0$,

(ii) $\operatorname{width} M = 0$, otherwise.

Next we state the proposition

(20) If $\operatorname{len} M > 0$, then for every n holds M is a matrix over D of dimension $\operatorname{len} M \times n$ iff $n = \operatorname{width} M$.

Let M be a tabular finite sequence. The indices of M yielding a set is defined as follows:

(Def. 5) The indices of $M = [: \operatorname{dom} M, \operatorname{Seg} \operatorname{width} M :]$.

Let D be a set, let M be a matrix over D, and let us consider i, j. Let us assume that $\langle i, j \rangle \in$ the indices of M. The functor $M \circ (i, j)$ yields an element of D and is defined as follows:

(Def. 6) There exists a finite sequence p of elements of D such that $p = M(i)$ and $M \circ (i, j) = p(j)$.

The following proposition is true

(21) If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_1 \circ (i, j) = M_2 \circ (i, j)$, then $M_1 = M_2$.

In this article we present several logical schemes. The scheme *MatrixLambda* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j if $\langle i, j\rangle \in$ the indices of M, then $M \circ (i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

The scheme *MatrixEx* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j if $\langle i, j\rangle \in$ the indices of M, then $\mathcal{P}[i, j, M \circ (i, j)]$

provided the parameters meet the following requirements:

- For all i, j such that $\langle i, j\rangle \in [:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:]$ and for all elements x_1, x_2 of $\mathcal{A}$ such that $\mathcal{P}[i, j, x_1]$ and $\mathcal{P}[i, j, x_2]$ holds $x_1 = x_2$, and
- For all i, j such that $\langle i, j\rangle \in [:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:]$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[i, j, x]$.

We now state several propositions:

(23)[1] For every matrix M over D of dimension $0 \times m$ holds $\operatorname{len} M = 0$ and $\operatorname{width} M = 0$ and the indices of $M = \emptyset$.

(24) Suppose $n > 0$. Let M be a matrix over D of dimension $n \times m$. Then $\operatorname{len} M = n$ and $\operatorname{width} M = m$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} m:]$.

(25) For every matrix M over D of dimension n holds $\operatorname{len} M = n$ and $\operatorname{width} M = n$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} n:]$.

(26) For every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} \operatorname{width} M:]$.

(27) For all matrices M_1, M_2 over D of dimension $n \times m$ holds the indices of $M_1 =$ the indices of M_2.

(28) Let M_1, M_2 be matrices over D of dimension $n \times m$. Suppose that for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_1 \circ (i, j) = M_2 \circ (i, j)$. Then $M_1 = M_2$.

(29) Let M_1 be a matrix over D of dimension n and given i, j. If $\langle i, j\rangle \in$ the indices of M_1, then $\langle j, i\rangle \in$ the indices of M_1.

Let us consider D and let M be a matrix over D. The functor M^{T} yielding a matrix over D is defined by the conditions (Def. 7).

(Def. 7)(i) $\operatorname{len}(M^{\mathrm{T}}) = \operatorname{width} M$,

(ii) for all i, j holds $\langle i, j\rangle \in$ the indices of M^{T} iff $\langle j, i\rangle \in$ the indices of M, and

(iii) for all i, j such that $\langle j, i\rangle \in$ the indices of M holds $M^{\mathrm{T}} \circ (i, j) = M \circ (j, i)$.

Let us consider D, M, i. The functor $\operatorname{Line}(M, i)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 8) $\operatorname{len}\operatorname{Line}(M, i) = \operatorname{width} M$ and for every j such that $j \in \operatorname{Seg}\operatorname{width} M$ holds $\operatorname{Line}(M, i)(j) = M \circ (i, j)$.

The functor $M_{\square, i}$ yields a finite sequence of elements of D and is defined by:

[1] The proposition (22) has been removed.

(Def. 9) $\mathrm{len}(M_{\square,i}) = \mathrm{len}M$ and for every j such that $j \in \mathrm{dom}M$ holds $M_{\square,i}(j) = M \circ (j,i)$.

Let us consider D, let M be a matrix over D, and let us consider i. Then $\mathrm{Line}(M,i)$ is an element of $D^{\mathrm{width}M}$. Then $M_{\square,i}$ is an element of $D^{\mathrm{len}M}$.

In the sequel A, B denote matrices over K of dimension n.

Let us consider K, n. The functor $K^{n\times n}$ yields a set and is defined as follows:

(Def. 10) $K^{n\times n} = ((\text{the carrier of } K)^n)^n$.

The functor $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n}$ yields a matrix over K of dimension n and is defined as follows:

(Def. 11) $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n} = n \mapsto (n \mapsto 0_K)$.

The functor $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ yielding a matrix over K of dimension n is defined by the conditions (Def. 12).

(Def. 12)(i) For every i such that $\langle i, i\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n} \circ (i,i) = \mathbf{1}_K$, and

(ii) for all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $i \neq j$ holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n} \circ (i,j) = 0_K$.

Let us consider A. The functor $-A$ yielding a matrix over K of dimension n is defined by:

(Def. 13) For all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(-A) \circ (i,j) = -(A \circ (i,j))$.

Let us consider B. The functor $A+B$ yields a matrix over K of dimension n and is defined as follows:

(Def. 14) For all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A+B) \circ (i,j) = (A \circ (i,j)) + (B \circ (i,j))$.

Let us consider K, n. Note that $K^{n\times n}$ is non empty.

We now state two propositions:

(30) If $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n}$, then $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n} \circ (i,j) = 0_K$.

(31) x is an element of $K^{n\times n}$ iff x is a matrix over K of dimension n.

Let us consider K, n. A matrix over K of dimension n is said to be a diagonal n-dimensional matrix over K if:

(Def. 15) For all i, j such that $\langle i, j\rangle \in$ the indices of it and it $\circ (i, j) \neq 0_K$ holds $i = j$.

In the sequel A, B, C denote matrices over F of dimension n.

One can prove the following propositions:

(32) $A + B = B + A$.

(33) $(A + B) + C = A + (B + C)$.

(34) $A + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n} = A$.

(35) $A + -A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

Let us consider F, n. The functor $F_{\mathrm{G}}^{n\times n}$ yields a strict Abelian group and is defined by:

(Def. 16) The carrier of $F_{\mathrm{G}}^{n\times n} = F^{n\times n}$ and for all A, B holds (the addition of $F_{\mathrm{G}}^{n\times n})(A, B) = A + B$ and the zero of $F_{\mathrm{G}}^{n\times n} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

Acknowledgments

I would like to thank Grzegorz Bancerek for his useful suggestions and valuable comments.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[10] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received June 8, 1991

Published January 2, 2004