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# Representation Theorem for Boolean Algebras

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The articles [13], [8], [15], [11], [16], [5], [7], [6], [4], [14], [12], [17], [2], [3], [9], [10], and [1] provide the notation and terminology for this paper.

In this paper $T$ denotes a non empty topological space and $X$ denotes a subset of $T$.

Let $T$ be a non empty topological structure. The functor OpenClosedSet($T$) yielding a family of subsets of $T$ is defined by:

(Def. 1) OpenClosedSet($T$) $= \{x; x$ ranges over subsets of $T$: $x$ is open and closed$\}$.

Let $T$ be a non empty topological space. Observe that OpenClosedSet($T$) is non empty.

We now state three propositions:

(2)[1] If $X \in$ OpenClosedSet($T$), then $X$ is open.

(3) If $X \in$ OpenClosedSet($T$), then $X$ is closed.

(4) If $X$ is open and closed, then $X \in$ OpenClosedSet($T$).

In the sequel $x$ denotes an element of OpenClosedSet($T$).

Let us consider $T$ and let $C$, $D$ be elements of OpenClosedSet($T$). Then $C \cup D$ is an element of OpenClosedSet($T$). Then $C \cap D$ is an element of OpenClosedSet($T$).

Let us consider $T$. The functor join($T$) yields a binary operation on OpenClosedSet($T$) and is defined as follows:

(Def. 2) For all elements $A$, $B$ of OpenClosedSet($T$) holds $(\mathrm{join}(T))(A, B) = A \cup B$.

The functor meet($T$) yielding a binary operation on OpenClosedSet($T$) is defined as follows:

(Def. 3) For all elements $A$, $B$ of OpenClosedSet($T$) holds $(\mathrm{meet}(T))(A, B) = A \cap B$.

One can prove the following propositions:

(5) For all elements $x$, $y$ of $\langle \mathrm{OpenClosedSet}(T), \mathrm{join}(T), \mathrm{meet}(T) \rangle$ and for all elements $x'$, $y'$ of OpenClosedSet($T$) such that $x = x'$ and $y = y'$ holds $x \sqcup y = x' \cup y'$.

(6) For all elements $x$, $y$ of $\langle \mathrm{OpenClosedSet}(T), \mathrm{join}(T), \mathrm{meet}(T) \rangle$ and for all elements $x'$, $y'$ of OpenClosedSet($T$) such that $x = x'$ and $y = y'$ holds $x \sqcap y = x' \cap y'$.

(7) $\emptyset_T$ is an element of OpenClosedSet($T$).

[1] The proposition (1) has been removed.



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(8) $\Omega_T$ is an element of OpenClosedSet$(T)$.

(9) $x^{\rm c}$ is an element of OpenClosedSet$(T)$.

(10) $\langle$OpenClosedSet$(T)$, join$(T)$, meet$(T)\rangle$ is a lattice.

Let $T$ be a non empty topological space. The functor OpenClosedSetLatt$(T)$ yields a lattice and is defined as follows:

(Def. 4) OpenClosedSetLatt$(T) = \langle$OpenClosedSet$(T)$, join$(T)$, meet$(T)\rangle$.

One can prove the following propositions:

(11) For every non empty topological space $T$ and for all elements $x$, $y$ of OpenClosedSetLatt$(T)$ holds $x \sqcup y = x \cup y$.

(12) For every non empty topological space $T$ and for all elements $x$, $y$ of OpenClosedSetLatt$(T)$ holds $x \sqcap y = x \cap y$.

(13) The carrier of OpenClosedSetLatt$(T)$ = OpenClosedSet$(T)$.

(14) OpenClosedSetLatt$(T)$ is Boolean.

(15) $\Omega_T$ is an element of OpenClosedSetLatt$(T)$.

(16) $\emptyset_T$ is an element of OpenClosedSetLatt$(T)$.

In the sequel $x$, $X$ denote sets.

Let us observe that there exists a Boolean lattice which is non trivial.

We adopt the following rules: $B_1$ denotes a non trivial Boolean lattice, $a$, $b$ denote elements of $B_1$, and $U_1$, $F$ denote filters of $B_1$.

Let us consider $B_1$. The functor ultraset$(B_1)$ yields a subset of $2^{\text{the carrier of } B_1}$ and is defined as follows:

(Def. 5) ultraset$(B_1) = \{F : F \text{ is an ultrafilter}\}$.

Let us consider $B_1$. Observe that ultraset$(B_1)$ is non empty.

The following propositions are true:

(18)[2] $x \in$ ultraset$(B_1)$ iff there exists $U_1$ such that $U_1 = x$ and $U_1$ is an ultrafilter.

(19) For every $a$ holds $\{F : F \text{ is an ultrafilter} \wedge a \in F\} \subseteq$ ultraset$(B_1)$.

Let us consider $B_1$. The functor UFilter$(B_1)$ yielding a function is defined by:

(Def. 6) dom UFilter$(B_1)$ = the carrier of $B_1$ and for every element $a$ of $B_1$ holds $(\text{UFilter}(B_1))(a) = \{U_1 : U_1 \text{ is an ultrafilter} \wedge a \in U_1\}$.

Next we state several propositions:

(20) $x \in (\text{UFilter}(B_1))(a)$ iff there exists $F$ such that $F = x$ and $F$ is an ultrafilter and $a \in F$.

(21) $F \in (\text{UFilter}(B_1))(a)$ iff $F$ is an ultrafilter and $a \in F$.

(22) For every $F$ such that $F$ is an ultrafilter holds $a \sqcup b \in F$ iff $a \in F$ or $b \in F$.

(23) $(\text{UFilter}(B_1))(a \sqcap b) = (\text{UFilter}(B_1))(a) \cap (\text{UFilter}(B_1))(b)$.

(24) $(\text{UFilter}(B_1))(a \sqcup b) = (\text{UFilter}(B_1))(a) \cup (\text{UFilter}(B_1))(b)$.

Let us consider $B_1$. Then UFilter$(B_1)$ is a function from the carrier of $B_1$ into $2^{\text{ultraset}(B_1)}$.

Let us consider $B_1$. The functor StoneR$(B_1)$ yields a set and is defined by:

[2] The proposition (17) has been removed.

(Def. 7) $\mathrm{StoneR}(B_1) = \mathrm{rng}\,\mathrm{UFilter}(B_1)$.

Let us consider $B_1$. One can check that $\mathrm{StoneR}(B_1)$ is non empty.

The following two propositions are true:

(25) $\mathrm{StoneR}(B_1) \subseteq 2^{\mathrm{ultraset}(B_1)}$.

(26) $x \in \mathrm{StoneR}(B_1)$ iff there exists $a$ such that $(\mathrm{UFilter}(B_1))(a) = x$.

Let us consider $B_1$. The functor $\mathrm{StoneSpace}(B_1)$ yielding a strict topological space is defined by:

(Def. 8) The carrier of $\mathrm{StoneSpace}(B_1) = \mathrm{ultraset}(B_1)$ and the topology of $\mathrm{StoneSpace}(B_1) = \{\bigcup A; A$ ranges over families of subsets of $\mathrm{ultraset}(B_1)$: $A \subseteq \mathrm{StoneR}(B_1)\}$.

Let us consider $B_1$. Note that $\mathrm{StoneSpace}(B_1)$ is non empty.

We now state two propositions:

(27) If $F$ is an ultrafilter and $F \notin (\mathrm{UFilter}(B_1))(a)$, then $a \notin F$.

(28) $\mathrm{ultraset}(B_1) \setminus (\mathrm{UFilter}(B_1))(a) = (\mathrm{UFilter}(B_1))(a^{\mathrm{c}})$.

Let us consider $B_1$. The functor $\mathrm{StoneBLattice}(B_1)$ yields a lattice and is defined by:

(Def. 9) $\mathrm{StoneBLattice}(B_1) = \mathrm{OpenClosedSetLatt}(\mathrm{StoneSpace}(B_1))$.

One can prove the following four propositions:

(29) $\mathrm{UFilter}(B_1)$ is one-to-one.

(30) $\bigcup \mathrm{StoneR}(B_1) = \mathrm{ultraset}(B_1)$.

(31) For all sets $A$, $B$, $X$ such that $X \subseteq \bigcup(A \cup B)$ and for every set $Y$ such that $Y \in B$ holds $Y$ misses $X$ holds $X \subseteq \bigcup A$.

(32) For every non empty set $X$ holds there exists a finite subset of $X$ which is non empty.

Let $D$ be a non empty set. One can verify that there exists a finite subset of $D$ which is non empty.

The following propositions are true:

(34)[3] Let $L$ be a non trivial Boolean lattice and $D$ be a non empty subset of $L$. Suppose $\bot_L \in [D)$. Then there exists a non empty finite subset $B$ of the carrier of $L$ such that $B \subseteq D$ and $\bigsqcap^{\mathrm{f}}_{B} = \bot_L$.

(35) For every lower bound lattice $L$ it is not true that there exists a filter $F$ of $L$ such that $F$ is an ultrafilter and $\bot_L \in F$.

(36) $(\mathrm{UFilter}(B_1))(\bot_{(B_1)}) = \emptyset$.

(37) $(\mathrm{UFilter}(B_1))(\top_{(B_1)}) = \mathrm{ultraset}(B_1)$.

(38) If $\mathrm{ultraset}(B_1) = \bigcup X$ and $X$ is a subset of $\mathrm{StoneR}(B_1)$, then there exists a finite subset $Y$ of $X$ such that $\mathrm{ultraset}(B_1) = \bigcup Y$.

(40)[4] $\mathrm{StoneR}(B_1) = \mathrm{OpenClosedSet}(\mathrm{StoneSpace}(B_1))$.

Let us consider $B_1$. Then $\mathrm{UFilter}(B_1)$ is a homomorphism from $B_1$ to $\mathrm{StoneBLattice}(B_1)$.

Next we state four propositions:

(41) $\mathrm{rng}\,\mathrm{UFilter}(B_1) =$ the carrier of $\mathrm{StoneBLattice}(B_1)$.

(42) $\mathrm{UFilter}(B_1)$ is isomorphism.

(43) $B_1$ and $\mathrm{StoneBLattice}(B_1)$ are isomorphic.

(44) For every non trivial Boolean lattice $B_1$ there exists a non empty topological space $T$ such that $B_1$ and $\mathrm{OpenClosedSetLatt}(T)$ are isomorphic.

[3] The proposition (33) has been removed.

[4] The proposition (39) has been removed.

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