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Banach Space of Bounded Linear Operators

Yasunari Shidama
Shinshu University
Nagano

Summary. On this article, the basic properties of linear spaces which are defined by the set of all linear operators from one linear space to another are described. Especially, the Banach space is introduced. This is defined by the set of all bounded linear operators.

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The articles [24], [6], [23], [31], [25], [33], [32], [4], [5], [16], [22], [21], [3], [1], [2], [28], [27], [9], [7], [30], [14], [26], [17], [29], [18], [19], [8], [20], [13], [11], [12], [10], and [15] provide the notation and terminology for this paper.

1. Real Vector Space of Operators

Let X be a set, let Y be a non empty set, let F be a function from $[:\mathbb{R}, Y:]$ into Y, let a be a real number, and let f be a function from X into Y. Then $F^\circ(a, f)$ is an element of Y^X.

One can prove the following two propositions:

(1) Let X be a non empty set and Y be a non empty loop structure. Then there exists a binary operation A_1 on (the carrier of $Y)^X$ such that for all elements f, g of (the carrier of $Y)^X$ holds $A_1(f, g) =$ (the addition of $Y)^\circ(f, g)$.

(2) Let X be a non empty set and Y be a real linear space. Then there exists a function M_1 from $[:\mathbb{R}$, (the carrier of $Y)^X:]$ into (the carrier of $Y)^X$ such that for every real number r and for every element f of (the carrier of $Y)^X$ and for every element s of X holds $M_1(\langle r, f\rangle)(s) = r \cdot f(s)$.

Let X be a non empty set and let Y be a non empty loop structure. The functor FuncAdd(X,Y) yielding a binary operation on (the carrier of $Y)^X$ is defined as follows:

(Def. 1) For all elements f, g of (the carrier of $Y)^X$ holds (FuncAdd$(X,Y))(f, g) =$ (the addition of $Y)^\circ(f, g)$.

Let X be a non empty set and let Y be a real linear space. The functor FuncExtMult(X,Y) yields a function from $[:\mathbb{R}$, (the carrier of $Y)^X:]$ into (the carrier of $Y)^X$ and is defined by the condition (Def. 2).

(Def. 2) Let a be a real number, f be an element of (the carrier of $Y)^X$, and x be an element of X. Then (FuncExtMult$(X,Y))(\langle a, f\rangle)(x) = a \cdot f(x)$.

Let X be a set and let Y be a non empty zero structure. The functor FuncZero(X,Y) yields an element of (the carrier of $Y)^X$ and is defined as follows:

(Def. 3) FuncZero$(X,Y) = X \longmapsto 0_Y$.

We adopt the following rules: X denotes a non empty set, Y denotes a real linear space, and f, g, h denote elements of (the carrier of $Y)^X$.

Next we state two propositions:

(3) Let Y be a non empty loop structure and f, g, h be elements of (the carrier of $Y)^X$. Then $h = (\mathrm{FuncAdd}(X,Y))(f, g)$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(4) For every element x of X holds $(\mathrm{FuncZero}(X,Y))(x) = 0_Y$.

In the sequel a, b are real numbers.

We now state several propositions:

(5) $h = (\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle)$ iff for every element x of X holds $h(x) = a \cdot f(x)$.

(6) $(\mathrm{FuncAdd}(X,Y))(f, g) = (\mathrm{FuncAdd}(X,Y))(g, f)$.

(7) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncAdd}(X,Y))(g, h)) = (\mathrm{FuncAdd}(X,Y))((\mathrm{FuncAdd}(X,Y))(f, g), h)$.

(8) $(\mathrm{FuncAdd}(X,Y))(\mathrm{FuncZero}(X,Y), f) = f$.

(9) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncExtMult}(X,Y))(\langle -1, f\rangle)) = \mathrm{FuncZero}(X,Y)$.

(10) $(\mathrm{FuncExtMult}(X,Y))(\langle 1, f\rangle) = f$.

(11) $(\mathrm{FuncExtMult}(X,Y))(\langle a, (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)\rangle) = (\mathrm{FuncExtMult}(X,Y))(\langle a \cdot b, f\rangle)$.

(12) $(\mathrm{FuncAdd}(X,Y))((\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle), (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)) = (\mathrm{FuncExtMult}(X,Y))(\langle a + b, f\rangle)$.

(13) $\langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$ is a real linear space.

Let X be a non empty set and let Y be a real linear space. The functor $\mathrm{RealVectSpace}(X,Y)$ yielding a real linear space is defined by:

(Def. 4) $\mathrm{RealVectSpace}(X,Y) = \langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$.

Let X be a non empty set and let Y be a real linear space. Note that $\mathrm{RealVectSpace}(X,Y)$ is strict.

Next we state three propositions:

(14) Let X be a non empty set, Y be a real linear space, f, g, h be vectors of $\mathrm{RealVectSpace}(X,Y)$, and f', g', h' be elements of (the carrier of $Y)^X$. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(15) Let X be a non empty set, Y be a real linear space, f, h be vectors of $\mathrm{RealVectSpace}(X,Y)$, and f', h' be elements of (the carrier of $Y)^X$. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(16) For every non empty set X and for every real linear space Y holds $0_{\mathrm{RealVectSpace}(X,Y)} = X \longmapsto 0_Y$.

2. Real Vector Space of Linear Operators

Let X be a non empty RLS structure, let Y be a non empty loop structure, and let I_1 be a function from X into Y. We say that I_1 is additive if and only if:

(Def. 5) For all vectors x, y of X holds $I_1(x+y) = I_1(x) + I_1(y)$.

Let X, Y be non empty RLS structures and let I_1 be a function from X into Y. We say that I_1 is homogeneous if and only if:

(Def. 6) For every vector x of X and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

Let X be a non empty RLS structure and let Y be a real linear space. Note that there exists a function from X into Y which is additive and homogeneous.

Let X, Y be real linear spaces. A linear operator from X into Y is an additive homogeneous function from X into Y.

Let X, Y be real linear spaces. The functor LinearOperators(X,Y) yields a subset of RealVectSpace(the carrier of X, Y) and is defined by:

(Def. 7) For every set x holds $x \in$ LinearOperators(X,Y) iff x is a linear operator from X into Y.

Let X, Y be real linear spaces. One can check that LinearOperators(X,Y) is non empty.

We now state two propositions:

(17) For all real linear spaces X, Y holds LinearOperators(X,Y) is linearly closed.

(18) Let X, Y be real linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is a subspace of RealVectSpace(the carrier of X, Y).

Let X, Y be real linear spaces. Note that $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The following proposition is true

(19) Let X, Y be real linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is a real linear space.

Let X, Y be real linear spaces. The functor RVectorSpaceOfLinearOperators(X,Y) yields a real linear space and is defined by:

(Def. 8) RVectorSpaceOfLinearOperators$(X,Y) = \langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$.

Let X, Y be real linear spaces. Note that RVectorSpaceOfLinearOperators(X,Y) is strict.

One can prove the following four propositions:

(20) Let X, Y be real linear spaces, f, g, h be vectors of RVectorSpaceOfLinearOperators(X,Y), and f', g', h' be linear operators from X into Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every vector x of X holds $h'(x) = f'(x) + g'(x)$.

(21) Let X, Y be real linear spaces, f, h be vectors of RVectorSpaceOfLinearOperators(X,Y), and f', h' be linear operators from X into Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h'(x) = a \cdot f'(x)$.

(22) For all real linear spaces X, Y holds $0_{\mathrm{RVectorSpaceOfLinearOperators}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

(23) For all real linear spaces X, Y holds (the carrier of X) $\longmapsto 0_Y$ is a linear operator from X into Y.

3. Real Normed Linear Space of Bounded Linear Operators

We now state the proposition

(24) Let X be a real normed space, s_1 be a sequence of X, and g be a point of X. If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim \|s_1\| = \|g\|$.

Let X, Y be real normed spaces and let I_1 be a linear operator from X into Y. We say that I_1 is bounded if and only if:

(Def. 9) There exists a real number K such that $0 \leq K$ and for every vector x of X holds $\|I_1(x)\| \leq K \cdot \|x\|$.

We now state the proposition

(25) Let X, Y be real normed spaces and f be a linear operator from X into Y. If for every vector x of X holds $f(x) = 0_Y$, then f is bounded.

Let X, Y be real normed spaces. Note that there exists a linear operator from X into Y which is bounded.

Let X, Y be real normed spaces. The functor BoundedLinearOperators(X,Y) yields a subset of RVectorSpaceOfLinearOperators(X,Y) and is defined as follows:

(Def. 10) For every set x holds $x \in$ BoundedLinearOperators(X,Y) iff x is a bounded linear operator from X into Y.

Let X, Y be real normed spaces. Note that BoundedLinearOperators(X,Y) is non empty.

One can prove the following propositions:

(26) For all real normed spaces X, Y holds BoundedLinearOperators(X,Y) is linearly closed.

(27) For all real normed spaces X, Y holds $\langle$BoundedLinearOperators(X,Y), Zero_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Add_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Mult_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y))$\rangle$ is a subspace of RVectorSpaceOfLinearOperators(X,Y).

Let X, Y be real normed spaces. Note that $\langle$BoundedLinearOperators(X,Y), Zero_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Add_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Mult_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

Next we state the proposition

(28) For all real normed spaces X, Y holds $\langle$BoundedLinearOperators(X,Y), Zero_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Add_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Mult_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y))$\rangle$ is a real linear space.

Let X, Y be real normed spaces. The functor RVectorSpaceOfBoundedLinearOperators(X,Y) yielding a real linear space is defined as follows:

(Def. 11) RVectorSpaceOfBoundedLinearOperators$(X,Y) = \langle$BoundedLinearOperators(X,Y), Zero_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Add_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y)), Mult_(BoundedLinearOperators(X,Y), RVectorSpaceOfLinearOperators(X,Y))$\rangle$.

Let X, Y be real normed spaces. Observe that RVectorSpaceOfBoundedLinearOperators(X,Y) is strict.

One can prove the following three propositions:

(29) Let X, Y be real normed spaces, f, g, h be vectors of RVectorSpaceOfBoundedLinearOperators(X,Y), and f', g', h' be bounded linear operators from X into Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every vector x of X holds $h'(x) = f'(x) + g'(x)$.

(30) Let X, Y be real normed spaces, f, h be vectors of RVectorSpaceOfBoundedLinearOperators(X,Y), and f', h' be bounded linear operators from X into Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h'(x) = a \cdot f'(x)$.

(31) For all real normed spaces X, Y holds $0_{\mathrm{RVectorSpaceOfBoundedLinearOperators}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

Let X, Y be real normed spaces and let f be a set. Let us assume that $f \in \mathrm{BoundedLinearOperators}(X,Y)$. The functor modetrans$(f,X,Y)$ yields a bounded linear operator from X into Y and is defined as follows:

(Def. 12) modetrans$(f,X,Y) = f$.

Let X, Y be real normed spaces and let u be a linear operator from X into Y. The functor PreNorms(u) yields a non empty subset of $\mathbb{R}$ and is defined by:

(Def. 13) $\mathrm{PreNorms}(u) = \{\|u(t)\|; t$ ranges over vectors of X: $\|t\| \leq 1\}$.

Next we state three propositions:

(32) Let X, Y be real normed spaces and g be a bounded linear operator from X into Y. Then PreNorms(g) is non empty and upper bounded.

(33) Let X, Y be real normed spaces and g be a linear operator from X into Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(34) Let X, Y be real normed spaces. Then there exists a function N_1 from BoundedLinearOperators(X,Y) into $\mathbb{R}$ such that for every set f if $f \in \mathrm{BoundedLinearOperators}(X,Y)$, then $N_1(f) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(f,X,Y))$.

Let X, Y be real normed spaces. The functor BoundedLinearOperatorsNorm(X,Y) yields a function from BoundedLinearOperators(X,Y) into $\mathbb{R}$ and is defined by:

(Def. 14) For every set x such that $x \in \mathrm{BoundedLinearOperators}(X,Y)$ holds $(\mathrm{BoundedLinearOperatorsNorm}(X,Y))(x) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(x,X,Y))$.

One can prove the following propositions:

(35) For all real normed spaces X, Y and for every bounded linear operator f from X into Y holds modetrans$(f,X,Y) = f$.

(36) For all real normed spaces X, Y and for every bounded linear operator f from X into Y holds $(\mathrm{BoundedLinearOperatorsNorm}(X,Y))(f) = \sup \mathrm{PreNorms}(f)$.

Let X, Y be real normed spaces. The functor RNormSpaceOfBoundedLinearOperators(X,Y) yielding a non empty normed structure is defined as follows:

(Def. 15) $\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y) = \langle \mathrm{BoundedLinearOperators}(X,Y), \mathrm{Zero\_}(\mathrm{BoundedLinearOperators}(X,Y), \mathrm{RVectorSpaceOfLinearOperators}(X,Y)), \mathrm{Add\_}(\mathrm{BoundedLinearOperators}(X,Y), \mathrm{RVectorSpaceOfLinearOperators}(X,Y)), \mathrm{Mult\_}(\mathrm{BoundedLinearOperators}(X,Y), \mathrm{RVectorSpaceOfLinearOperators}(X,Y)), \mathrm{BoundedLinearOperatorsNorm}(X,Y) \rangle$.

We now state several propositions:

(37) For all real normed spaces X, Y holds (the carrier of X) $\longmapsto 0_Y = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)}$.

(38) Let X, Y be real normed spaces, f be a point of RNormSpaceOfBoundedLinearOperators(X,Y), and g be a bounded linear operator from X into Y. If $g = f$, then for every vector t of X holds $\|g(t)\| \leq \|f\| \cdot \|t\|$.

(39) For all real normed spaces X, Y and for every point f of RNormSpaceOfBoundedLinearOperators(X,Y) holds $0 \leq \|f\|$.

(40) For all real normed spaces X, Y and for every point f of RNormSpaceOfBoundedLinearOperators(X,Y) such that $f = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)}$ holds $0 = \|f\|$.

(41) Let X, Y be real normed spaces, f, g, h be points of RNormSpaceOfBoundedLinearOperators(X,Y), and f', g', h' be bounded linear operators from X into Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every vector x of X holds $h'(x) = f'(x) + g'(x)$.

(42) Let X, Y be real normed spaces, f, h be points of RNormSpaceOfBoundedLinearOperators(X,Y), and f', h' be bounded linear operators from X into Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h'(x) = a \cdot f'(x)$.

(43) Let X be a real normed space, Y be a real normed space, f, g be points of RNormSpaceOfBoundedLinearOperators(X,Y), and a be a real number. Then $\|f\| = 0$ iff $f = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)}$ and $\|a \cdot f\| = |a| \cdot \|f\|$ and $\|f+g\| \leq \|f\| + \|g\|$.

(44) For all real normed spaces X, Y holds RNormSpaceOfBoundedLinearOperators(X,Y) is real normed space-like.

(45) For all real normed spaces X, Y holds RNormSpaceOfBoundedLinearOperators(X,Y) is a real normed space.

Let X, Y be real normed spaces. Observe that RNormSpaceOfBoundedLinearOperators(X,Y) is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

The following proposition is true

(46) Let X, Y be real normed spaces, f, g, h be points of RNormSpaceOfBoundedLinearOperators(X,Y), and f', g', h' be bounded linear operators from X into Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every vector x of X holds $h'(x) = f'(x) - g'(x)$.

4. Real Banach Space of Bounded Linear Operators

Let X be a real normed space. We say that X is complete if and only if:

(Def. 16) For every sequence s_1 of X such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

One can verify that there exists a real normed space which is complete.

A real Banach space is a complete real normed space.

Next we state three propositions:

(47) Let X be a real normed space and s_1 be a sequence of X. If s_1 is convergent, then $\|s_1\|$ is convergent and $\lim \|s_1\| = \|\lim s_1\|$.

(48) Let X, Y be real normed spaces. Suppose Y is complete. Let s_1 be a sequence of RNormSpaceOfBoundedLinearOperators(X,Y). If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(49) For every real normed space X and for every real Banach space Y holds RNormSpaceOfBoundedLinearOperators(X,Y) is a real Banach space.

Let X be a real normed space and let Y be a real Banach space. Observe that RNormSpaceOfBoundedLinearOperators(X,Y) is complete.

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