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Transitive Closure of Fuzzy Relations[1]

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The articles [?], and [?] provide the notation and terminology for this paper.

1. Inclusion of Fuzzy Sets

In this paper X, Y are non empty sets.

Let X be a non empty set. Observe that every membership function of X is real-yielding.

Let f, g be real-yielding functions. The predicate $f \sqsubseteq g$ is defined by:

(Def. 1) $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \leq g(x)$.

Let X be a non empty set and let f, g be membership functions of X. Let us observe that $f \sqsubseteq g$ if and only if:

(Def. 2) For every element x of X holds $f(x) \leq g(x)$.

We introduce $f \subseteq g$ as a synonym of $f \sqsubseteq g$.

Let X, Y be non empty sets and let f, g be membership functions of X, Y. Let us observe that $f \sqsubseteq g$ if and only if:

(Def. 3) For every element x of X and for every element y of Y holds $f(\langle x, y\rangle) \leq g(\langle x, y\rangle)$.

Next we state several propositions:

(1) For all membership functions R, S of X such that for every element x of X holds $R(x) = S(x)$ holds $R = S$.

(2) Let R, S be membership functions of X, Y. Suppose that for every element x of X and for every element y of Y holds $R(\langle x, y\rangle) = S(\langle x, y\rangle)$. Then $R = S$.

(3) For all membership functions R, S of X holds $R = S$ iff $R \subseteq S$ and $S \subseteq R$.

(4) For every membership function R of X holds $R \subseteq R$.

(5) For all membership functions R, S, T of X such that $R \subseteq S$ and $S \subseteq T$ holds $R \subseteq T$.

(6) Let X, Y, Z be non empty sets, R, S be membership functions of X, Y, and T, U be membership functions of Y, Z. If $R \subseteq S$ and $T \subseteq U$, then $R\,T \subseteq S\,U$.

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Let X be a non empty set and let f, g be membership functions of X. Let us notice that the functor $\min(f,g)$ is commutative. Let us observe that the functor $\max(f,g)$ is commutative.

We now state two propositions:

(7) For all membership functions f, g of X holds $\min(f,g) \subseteq f$.

(8) For all membership functions f, g of X holds $f \subseteq \max(f,g)$.

2. Properties of Fuzzy Relations

Let X be a non empty set and let R be a membership function of X, X. We say that R is reflexive if and only if:

(Def. 4) $\mathrm{Imf}(X,X) \subseteq R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is reflexive if and only if:

(Def. 5) For every element x of X holds $R(\langle x, x\rangle) = 1$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is symmetric if and only if:

(Def. 6) $\mathrm{converse} R = R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is symmetric if and only if:

(Def. 7) For all elements x, y of X holds $R(\langle x, y\rangle) = R(\langle y, x\rangle)$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is transitive if and only if:

(Def. 8) $R\,R \subseteq R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is transitive if and only if:

(Def. 9) For all elements x, y, z of X holds $R(\langle x, y\rangle) \sqcap R(\langle y, z\rangle) \preceq R(\langle x, z\rangle)$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is antisymmetric if and only if:

(Def. 10) For all elements x, y of X such that $R(\langle x, y\rangle) \neq 0$ and $R(\langle y, x\rangle) \neq 0$ holds $x = y$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is antisymmetric if and only if:

(Def. 11) For all elements x, y of X such that $R(\langle x, y\rangle) \neq 0$ and $x \neq y$ holds $R(\langle y, x\rangle) = 0$.

Let us consider X. One can check that $\mathrm{Imf}(X,X)$ is symmetric, transitive, reflexive, and antisymmetric.

Let us consider X. Observe that there exists a membership function of X, X which is reflexive, transitive, symmetric, and antisymmetric.

We now state two propositions:

(9) For all membership functions R, S of X, X such that R is symmetric and S is symmetric holds $\mathrm{converse}\min(R,S) = \min(R,S)$.

(10) For all membership functions R, S of X, X such that R is symmetric and S is symmetric holds $\mathrm{converse}\max(R,S) = \max(R,S)$.