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Introduction to Lattice Theory

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Summary. A lattice is defined as an algebra on a nonempty set with binary operations join and meet which are commutative and associative, and satisfy the absorption identities. The following kinds of lattices are considered: distributive, modular, bounded (with zero and unit elements), complemented, and Boolean (with complement). The article includes also theorems which immediately follow from definitions.

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The articles [2], [3], and [1] provide the notation and terminology for this paper.

We introduce $\sqcap$-semi lattice structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, a meet operation ⟩,

where the carrier is a set and the meet operation is a binary operation on the carrier.

We introduce $\sqcup$-semi lattice structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, a join operation ⟩,

where the carrier is a set and the join operation is a binary operation on the carrier.

We consider lattice structures as extensions of $\sqcap$-semi lattice structure and $\sqcup$-semi lattice structure as systems

⟨ a carrier, a join operation, a meet operation ⟩,

where the carrier is a set and the join operation and the meet operation are binary operations on the carrier.

One can check the following observations:

* there exists a $\sqcup$-semi lattice structure which is strict and non empty,
* there exists a $\sqcap$-semi lattice structure which is strict and non empty, and
* there exists a lattice structure which is strict and non empty.

Let G be a non empty $\sqcup$-semi lattice structure and let p, q be elements of G. The functor $p \sqcup q$ yields an element of G and is defined as follows:

(Def. 1) $p \sqcup q = (\text{the join operation of } G)(p, q)$.

Let G be a non empty $\sqcap$-semi lattice structure and let p, q be elements of G. The functor $p \sqcap q$ yielding an element of G is defined as follows:

(Def. 2) $p \sqcap q = (\text{the meet operation of } G)(p, q)$.

Let G be a non empty $\sqcup$-semi lattice structure and let p, q be elements of G. The predicate $p \sqsubseteq q$ is defined as follows:

(Def. 3) $p \sqcup q = q$.

Let I_1 be a non empty $\sqcup$-semi lattice structure. We say that I_1 is join-commutative if and only if:

(Def. 4) For all elements a, b of I_1 holds $a \sqcup b = b \sqcup a$.

We say that I_1 is join-associative if and only if:

(Def. 5) For all elements a, b, c of I_1 holds $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$.

Let I_1 be a non empty $\sqcap$-semi lattice structure. We say that I_1 is meet-commutative if and only if:

(Def. 6) For all elements a, b of I_1 holds $a \sqcap b = b \sqcap a$.

We say that I_1 is meet-associative if and only if:

(Def. 7) For all elements a, b, c of I_1 holds $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$.

Let I_1 be a non empty lattice structure. We say that I_1 is meet-absorbing if and only if:

(Def. 8) For all elements a, b of I_1 holds $(a \sqcap b) \sqcup b = b$.

We say that I_1 is join-absorbing if and only if:

(Def. 9) For all elements a, b of I_1 holds $a \sqcap (a \sqcup b) = a$.

Let I_1 be a non empty lattice structure. We say that I_1 is lattice-like if and only if the condition (Def. 10) is satisfied.

(Def. 10) I_1 is join-commutative, join-associative, meet-absorbing, meet-commutative, meet-associative, and join-absorbing.

Let us note that every non empty lattice structure which is lattice-like is also join-commutative, join-associative, meet-absorbing, meet-commutative, meet-associative, and join-absorbing and every non empty lattice structure which is join-commutative, join-associative, meet-absorbing, meet-commutative, meet-associative, and join-absorbing is also lattice-like.

One can verify the following observations:

* there exists a non empty $\sqcup$-semi lattice structure which is strict, join-commutative, and join-associative,
* there exists a non empty $\sqcap$-semi lattice structure which is strict, meet-commutative, and meet-associative, and
* there exists a non empty lattice structure which is strict and lattice-like.

A lattice is a lattice-like non empty lattice structure.

Let L be a join-commutative non empty $\sqcup$-semi lattice structure and let a, b be elements of L. Let us note that the functor $a \sqcup b$ is commutative.

Let L be a meet-commutative non empty $\sqcap$-semi lattice structure and let a, b be elements of L. Let us observe that the functor $a \sqcap b$ is commutative.

Let I_1 be a non empty lattice structure. We say that I_1 is distributive if and only if:

(Def. 11) For all elements a, b, c of I_1 holds $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

Let I_1 be a non empty lattice structure. We say that I_1 is modular if and only if:

(Def. 12) For all elements a, b, c of I_1 such that $a \sqsubseteq c$ holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$.

Let I_1 be a non empty $\sqcap$-semi lattice structure. We say that I_1 is lower-bounded if and only if:

(Def. 13) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcap a = c$ and $a \sqcap c = c$.

Let I_1 be a non empty $\sqcup$-semi lattice structure. We say that I_1 is upper-bounded if and only if:

(Def. 14) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcup a = c$ and $a \sqcup c = c$.

Let us observe that there exists a lattice which is strict, distributive, lower-bounded, upper-bounded, and modular.

A distributive lattice is a distributive lattice. A modular lattice is a modular lattice. A lower bound lattice is a lower-bounded lattice. An upper bound lattice is an upper-bounded lattice.

Let I_1 be a non empty lattice structure. We say that I_1 is bounded if and only if:

(Def. 15) I_1 is lower-bounded and upper-bounded.

Let us observe that every non empty lattice structure which is lower-bounded and upper-bounded is also bounded and every non empty lattice structure which is bounded is also lower-bounded and upper-bounded.

Let us note that there exists a lattice which is bounded and strict.

A bound lattice is a bounded lattice.

Let L be a non empty $\sqcap$-semi lattice structure. Let us assume that L is lower-bounded. The functor $\perp_L$ yields an element of L and is defined by:

(Def. 16) For every element a of L holds $\perp_L \sqcap a = \perp_L$ and $a \sqcap \perp_L = \perp_L$.

Let L be a non empty $\sqcup$-semi lattice structure. Let us assume that L is upper-bounded. The functor $\top_L$ yields an element of L and is defined as follows:

(Def. 17) For every element a of L holds $\top_L \sqcup a = \top_L$ and $a \sqcup \top_L = \top_L$.

Let L be a non empty lattice structure and let a, b be elements of L. We say that a is a complement of b if and only if:

(Def. 18) $a \sqcup b = \top_L$ and $b \sqcup a = \top_L$ and $a \sqcap b = \perp_L$ and $b \sqcap a = \perp_L$.

Let I_1 be a non empty lattice structure. We say that I_1 is complemented if and only if:

(Def. 19) For every element b of I_1 holds there exists an element of I_1 which is a complement of b.

Let us note that there exists a lattice which is bounded, complemented, and strict.

A complemented lattice is a complemented bound lattice.

Let I_1 be a non empty lattice structure. We say that I_1 is Boolean if and only if:

(Def. 20) I_1 is bounded, complemented, and distributive.

Let us mention that every non empty lattice structure which is Boolean is also bounded, complemented, and distributive and every non empty lattice structure which is bounded, complemented, and distributive is also Boolean.

Let us observe that there exists a lattice which is Boolean and strict.

A Boolean lattice is a Boolean lattice.

In the sequel L denotes a meet-absorbing join-absorbing meet-commutative non empty lattice structure and a denotes an element of L.

Next we state two propositions:

(17)[1] $a \sqcup a = a$.

(18) $a \sqcap a = a$.

In the sequel L denotes a lattice and a, b, c denote elements of L.

One can prove the following propositions:

(19) For all a, b, c holds $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ iff for all a, b, c holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

[1] The propositions (1)–(16) have been removed.

(21)[2] Let L be a meet-absorbing join-absorbing non empty lattice structure and a, b be elements of L. Then $a \sqsubseteq b$ if and only if $a \sqcap b = a$.

(22) Let L be a meet-absorbing join-absorbing join-associative meet-commutative non empty lattice structure and a, b be elements of L. Then $a \sqsubseteq a \sqcup b$.

(23) For every meet-absorbing meet-commutative non empty lattice structure L and for all elements a, b of L holds $a \sqcap b \sqsubseteq a$.

Let L be a meet-absorbing join-absorbing meet-commutative non empty lattice structure and let a, b be elements of L. Let us note that the predicate $a \sqsubseteq b$ is reflexive.

We now state four propositions:

(25)[3] Let L be a join-associative non empty $\sqcup$-semi lattice structure and a, b, c be elements of L. If $a \sqsubseteq b$ and $b \sqsubseteq c$, then $a \sqsubseteq c$.

(26) Let L be a join-commutative non empty $\sqcup$-semi lattice structure and a, b be elements of L. If $a \sqsubseteq b$ and $b \sqsubseteq a$, then $a = b$.

(27) Let L be a meet-absorbing join-absorbing meet-associative non empty lattice structure and a, b, c be elements of L. If $a \sqsubseteq b$, then $a \sqcap c \sqsubseteq b \sqcap c$.

(29)[4] For every lattice L such that for all elements a, b, c of L holds $(a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a) = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)$ holds L is distributive.

In the sequel L is a distributive lattice and a, b, c are elements of L.

The following propositions are true:

(31)[5] $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

(32) If $c \sqcap a = c \sqcap b$ and $c \sqcup a = c \sqcup b$, then $a = b$.

(34)[6] $(a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a) = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a)$.

One can check that every lattice which is distributive is also modular.

In the sequel L denotes a lower bound lattice and a denotes an element of L.

Next we state three propositions:

(39)[7] $\perp_L \sqcup a = a$.

(40) $\perp_L \sqcap a = \perp_L$.

(41) $\perp_L \sqsubseteq a$.

In the sequel L denotes an upper bound lattice and a denotes an element of L.

We now state three propositions:

(43)[8] $\top_L \sqcap a = a$.

(44) $\top_L \sqcup a = \top_L$.

(45) $a \sqsubseteq \top_L$.

Let L be a non empty lattice structure and let x be an element of L. Let us assume that L is a complemented distributive lattice. The functor x^{c} yielding an element of L is defined as follows:

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[2] The proposition (20) has been removed.
[3] The proposition (24) has been removed.
[4] The proposition (28) has been removed.
[5] The proposition (30) has been removed.
[6] The proposition (33) has been removed.
[7] The propositions (35)–(38) have been removed.
[8] The proposition (42) has been removed.

(Def. 21) x^{c} is a complement of x.

In the sequel L is a Boolean lattice and a, b are elements of L.
One can prove the following propositions:

(47)[9] $a^{c} \sqcap a = \perp_L$.

(48) $a^{c} \sqcup a = \top_L$.

(49) $(a^{c})^{c} = a$.

(50) $(a \sqcap b)^{c} = a^{c} \sqcup b^{c}$.

(51) $(a \sqcup b)^{c} = a^{c} \sqcap b^{c}$.

(52) $b \sqcap a = \perp_L$ iff $b \sqsubseteq a^{c}$.

(53) If $a \sqsubseteq b$, then $b^{c} \sqsubseteq a^{c}$.

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[3] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

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[9] The proposition (46) has been removed.