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Fixpoints in Complete Lattices[1]

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Summary. Theorem (5) states that if an iterate of a function has a unique fixpoint then it is also the fixpoint of the function. It has been included here in response to P. Andrews claim that such a proof in set theory takes thousands of lines when one starts with the axioms. While probably true, such a claim is misleading about the usefulness of proof-checking systems based on set theory.

Next, we prove the existence of the least and the greatest fixpoints for $\subseteq$-monotone functions from a powerset to a powerset of a set. Scheme *Knaster* is the Knaster theorem about the existence of fixpoints, cf. [13]. Theorem (11) is the Banach decomposition theorem which is then used to prove the Schröder-Bernstein theorem (12) (we followed Paulson's development of these theorems in Isabelle [15]). It is interesting to note that the last theorem when stated in Mizar in terms of cardinals becomes trivial to prove as in the Mizar development of cardinals the $\leq$ relation is synonymous with $\subseteq$.

Section 3 introduces the notion of the lattice of a lattice subset provided the subset has lubs and glbs.

The main theorem of Section 4 is the Tarski theorem (43) that every monotone function f over a complete lattice L has a complete lattice of fixpoints. As the consequence of this theorem we get the existence of the least fixpoint equal to $f^{\beta}(\perp_L)$ for some ordinal β with cardinality not bigger than the cardinality of the carrier of L, cf. [13], and analogously the existence of the greatest fixpoint equal to $f^{\beta}(\top_L)$.

Section 5 connects the fixpoint properties of monotone functions over complete lattices with the fixpoints of $\subseteq$-monotone functions over the lattice of subsets of a set (Boolean lattice).

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The articles [17], [11], [19], [20], [22], [21], [8], [10], [9], [16], [14], [12], [2], [3], [1], [6], [23], [4], [5], [18], and [7] provide the notation and terminology for this paper.

1. Preliminaries

In this paper f, g, h denote functions.

One can prove the following two propositions:

(1) If f is one-to-one and g is one-to-one and $\operatorname{rng} f$ misses $\operatorname{rng} g$, then $f + \cdot g$ is one-to-one.

(3)[1] Suppose $h = f \cup g$ and $\operatorname{dom} f$ misses $\operatorname{dom} g$. Then h is one-to-one if and only if the following conditions are satisfied:

(i) f is one-to-one,

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[1] The proposition (2) has been removed.

(ii) g is one-to-one, and

(iii) $\operatorname{rng} f$ misses $\operatorname{rng} g$.

2. Fixpoints in General

Let x be a set and let f be a function. We say that x is a fixpoint of f if and only if:

(Def. 1) $x \in \operatorname{dom} f$ and $x = f(x)$.

Let A be a non empty set, let a be an element of A, and let f be a function from A into A. Let us observe that a is a fixpoint of f if and only if:

(Def. 2) $a = f(a)$.

For simplicity, we adopt the following convention: x, y, X are sets, A is a non empty set, n is a natural number, and f is a function from X into X.

We now state two propositions:

(4) If x is a fixpoint of f^n, then $f(x)$ is a fixpoint of f^n.

(5) If there exists n such that x is a fixpoint of f^n and for every y such that y is a fixpoint of f^n holds $x = y$, then x is a fixpoint of f.

Let A, B be non empty sets and let f be a function from A into B. Let us observe that f is $\subseteq$-monotone if and only if:

(Def. 3) For all elements x, y of A such that $x \subseteq y$ holds $f(x) \subseteq f(y)$.

Let A be a set and let B be a non empty set. Observe that there exists a function from A into B which is $\subseteq$-monotone.

Let X be a set and let f be a $\subseteq$-monotone function from 2^X into 2^X. The functor $\operatorname{lfp}(X, f)$ yielding a subset of X is defined by:

(Def. 4) $\operatorname{lfp}(X, f) = \bigcap\{h; h$ ranges over subsets of X: $f(h) \subseteq h\}$.

The functor $\operatorname{gfp}(X, f)$ yields a subset of X and is defined as follows:

(Def. 5) $\operatorname{gfp}(X, f) = \bigcup\{h; h$ ranges over subsets of X: $h \subseteq f(h)\}$.

In the sequel f denotes a $\subseteq$-monotone function from 2^X into 2^X and S denotes a subset of X.

The following propositions are true:

(6) $\operatorname{lfp}(X, f)$ is a fixpoint of f.

(7) $\operatorname{gfp}(X, f)$ is a fixpoint of f.

(8) If $f(S) \subseteq S$, then $\operatorname{lfp}(X, f) \subseteq S$.

(9) If $S \subseteq f(S)$, then $S \subseteq \operatorname{gfp}(X, f)$.

(10) If S is a fixpoint of f, then $\operatorname{lfp}(X, f) \subseteq S$ and $S \subseteq \operatorname{gfp}(X, f)$.

The scheme *Knaster* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a set D such that $\mathcal{F}(D) = D$ and $D \subseteq \mathcal{A}$

provided the parameters meet the following conditions:

- For all sets X, Y such that $X \subseteq Y$ holds $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$, and
- $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}$.

In the sequel X, Y are non empty sets, f is a function from X into Y, and g is a function from Y into X.

The following four propositions are true:

(11) There exist sets X_1, X_2, Y_1, Y_2 such that X_1 misses X_2 and Y_1 misses Y_2 and $X_1 \cup X_2 = X$ and $Y_1 \cup Y_2 = Y$ and $f^\circ X_1 = Y_1$ and $g^\circ Y_2 = X_2$.

(12) If f is one-to-one and g is one-to-one, then there exists a function from X into Y which is bijective.

(13) If there exists f which is bijective, then $X \approx Y$.

(14) If f is one-to-one and g is one-to-one, then $X \approx Y$.

3. THE LATTICE OF LATTICE SUBSET

Let L be a non empty lattice structure, let f be a unary operation on L, and let x be an element of L. Then $f(x)$ is an element of L.

Let L be a lattice, let f be a function from the carrier of L into the carrier of L, let x be an element of L, and let O be an ordinal number. The functor $f^O_\sqcup(x)$ is defined by the condition (Def. 6).

(Def. 6) There exists a transfinite sequence L_0 such that

(i) $f^O_\sqcup(x) = \operatorname{last} L_0$,

(ii) $\operatorname{dom} L_0 = \operatorname{succ} O$,

(iii) $L_0(\emptyset) = x$,

(iv) for every ordinal number C such that $\operatorname{succ} C \in \operatorname{succ} O$ holds $L_0(\operatorname{succ} C) = f(L_0(C))$, and

(v) for every ordinal number C such that $C \in \operatorname{succ} O$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L_0(C) = \bigsqcup_L \operatorname{rng}(L_0 \restriction C)$.

The functor $f^O_\sqcap(x)$ is defined by the condition (Def. 7).

(Def. 7) There exists a transfinite sequence L_0 such that

(i) $f^O_\sqcap(x) = \operatorname{last} L_0$,

(ii) $\operatorname{dom} L_0 = \operatorname{succ} O$,

(iii) $L_0(\emptyset) = x$,

(iv) for every ordinal number C such that $\operatorname{succ} C \in \operatorname{succ} O$ holds $L_0(\operatorname{succ} C) = f(L_0(C))$, and

(v) for every ordinal number C such that $C \in \operatorname{succ} O$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L_0(C) = \bigsqcap_L \operatorname{rng}(L_0 \restriction C)$.

For simplicity, we adopt the following rules: L denotes a lattice, f denotes a function from the carrier of L into the carrier of L, x denotes an element of L, O, O_1, O_2 denote ordinal numbers, and T denotes a transfinite sequence.

We now state several propositions:

(16)[2] $f^\emptyset_\sqcup(x) = x$.

(17) $f^\emptyset_\sqcap(x) = x$.

(18) $f^{\operatorname{succ} O}_\sqcup(x) = f(f^O_\sqcup(x))$.

(19) $f^{\operatorname{succ} O}_\sqcap(x) = f(f^O_\sqcap(x))$.

(20) Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\operatorname{dom} T = O$ and for every ordinal number A such that $A \in O$ holds $T(A) = f^A_\sqcup(x)$. Then $f^O_\sqcup(x) = \bigsqcup_L \operatorname{rng} T$.

(21) Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\operatorname{dom} T = O$ and for every ordinal number A such that $A \in O$ holds $T(A) = f^A_\sqcap(x)$. Then $f^O_\sqcap(x) = \bigsqcap_L \operatorname{rng} T$.

(22) $f^n(x) = f^n_\sqcup(x)$.

[2] The proposition (15) has been removed.

(23) $f^n(x) = f^n_{\sqcap}(x)$.

Let L be a lattice, let f be a unary operation on the carrier of L, let a be an element of L, and let O be an ordinal number. Then $f^O_{\sqcup}(a)$ is an element of L.

Let L be a lattice, let f be a unary operation on the carrier of L, let a be an element of L, and let O be an ordinal number. Then $f^O_{\sqcap}(a)$ is an element of L.

Let L be a non empty lattice structure and let P be a subset of L. We say that P has l.u.b.'s if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let x, y be elements of L. Suppose $x \in P$ and $y \in P$. Then there exists an element z of L such that $z \in P$ and $x \sqsubseteq z$ and $y \sqsubseteq z$ and for every element z' of L such that $z' \in P$ and $x \sqsubseteq z'$ and $y \sqsubseteq z'$ holds $z \sqsubseteq z'$.

We say that P has g.l.b.'s if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x, y be elements of L. Suppose $x \in P$ and $y \in P$. Then there exists an element z of L such that $z \in P$ and $z \sqsubseteq x$ and $z \sqsubseteq y$ and for every element z' of L such that $z' \in P$ and $z' \sqsubseteq x$ and $z' \sqsubseteq y$ holds $z' \sqsubseteq z$.

Let L be a lattice. Note that there exists a subset of L which is non empty and has l.u.b.'s and g.l.b.'s.

Let L be a lattice and let P be a non empty subset of L with l.u.b.'s and g.l.b.'s. The functor $\mathbb{L}_P$ yielding a strict lattice is defined by the conditions (Def. 10).

(Def. 10)(i) The carrier of $\mathbb{L}_P = P$, and

(ii) for all elements x, y of $\mathbb{L}_P$ there exist elements x', y' of L such that $x = x'$ and $y = y'$ and $x \sqsubseteq y$ iff $x' \sqsubseteq y'$.

4. Complete lattices

Let us mention that every lattice which is complete is also bounded.

In the sequel L is a complete lattice, f is a monotone unary operation on L, and a, b are elements of L.

We now state a number of propositions:

(24) There exists a which is a fixpoint of f.

(25) For every a such that $a \sqsubseteq f(a)$ and for every O holds $a \sqsubseteq f^O_{\sqcup}(a)$.

(26) For every a such that $f(a) \sqsubseteq a$ and for every O holds $f^O_{\sqcap}(a) \sqsubseteq a$.

(27) For every a such that $a \sqsubseteq f(a)$ and for all O_1, O_2 such that $O_1 \subseteq O_2$ holds $f^{O_1}_{\sqcup}(a) \sqsubseteq f^{O_2}_{\sqcup}(a)$.

(28) For every a such that $f(a) \sqsubseteq a$ and for all O_1, O_2 such that $O_1 \subseteq O_2$ holds $f^{O_2}_{\sqcap}(a) \sqsubseteq f^{O_1}_{\sqcap}(a)$.

(29) For every a such that $a \sqsubseteq f(a)$ and for all O_1, O_2 such that $O_1 \subset O_2$ and $f^{O_2}_{\sqcup}(a)$ is not a fixpoint of f holds $f^{O_1}_{\sqcup}(a) \neq f^{O_2}_{\sqcup}(a)$.

(30) For every a such that $f(a) \sqsubseteq a$ and for all O_1, O_2 such that $O_1 \subset O_2$ and $f^{O_2}_{\sqcap}(a)$ is not a fixpoint of f holds $f^{O_1}_{\sqcap}(a) \neq f^{O_2}_{\sqcap}(a)$.

(31) If $a \sqsubseteq f(a)$ and $f^{O_1}_{\sqcup}(a)$ is a fixpoint of f, then for every O_2 such that $O_1 \subseteq O_2$ holds $f^{O_1}_{\sqcup}(a) = f^{O_2}_{\sqcup}(a)$.

(32) If $f(a) \sqsubseteq a$ and $f^{O_1}_{\sqcap}(a)$ is a fixpoint of f, then for every O_2 such that $O_1 \subseteq O_2$ holds $f^{O_1}_{\sqcap}(a) = f^{O_2}_{\sqcap}(a)$.

(33) For every a such that $a \sqsubseteq f(a)$ there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^O_{\sqcup}(a)$ is a fixpoint of f.

(34) For every a such that $f(a) \sqsubseteq a$ there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^{O}_{\sqcap}(a)$ is a fixpoint of f.

(35) Let given a, b. Suppose a is a fixpoint of f and b is a fixpoint of f. Then there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^{O}_{\sqcup}(a \sqcup b)$ is a fixpoint of f and $a \sqsubseteq f^{O}_{\sqcup}(a \sqcup b)$ and $b \sqsubseteq f^{O}_{\sqcup}(a \sqcup b)$.

(36) Let given a, b. Suppose a is a fixpoint of f and b is a fixpoint of f. Then there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^{O}_{\sqcap}(a \sqcap b)$ is a fixpoint of f and $f^{O}_{\sqcap}(a \sqcap b) \sqsubseteq a$ and $f^{O}_{\sqcap}(a \sqcap b) \sqsubseteq b$.

(37) If $a \sqsubseteq f(a)$ and $a \sqsubseteq b$ and b is a fixpoint of f, then for every O_2 holds $f^{O_2}_{\sqcup}(a) \sqsubseteq b$.

(38) If $f(a) \sqsubseteq a$ and $b \sqsubseteq a$ and b is a fixpoint of f, then for every O_2 holds $b \sqsubseteq f^{O_2}_{\sqcap}(a)$.

Let L be a complete lattice and let f be a unary operation on L. Let us assume that f is monotone. The functor FixPoints(f) yields a strict lattice and is defined as follows:

(Def. 11) There exists a non empty subset P of L with l.u.b.'s and g.l.b.'s such that $P = \{x; x$ ranges over elements of L: x is a fixpoint of $f\}$ and FixPoints$(f) = \mathbb{L}_P$.

The following propositions are true:

(39) The carrier of FixPoints$(f) = \{x; x$ ranges over elements of L: x is a fixpoint of $f\}$.

(40) The carrier of FixPoints$(f) \subseteq$ the carrier of L.

(41) $a \in$ the carrier of FixPoints(f) iff a is a fixpoint of f.

(42) For all elements x, y of FixPoints(f) and for all a, b such that $x = a$ and $y = b$ holds $x \sqsubseteq y$ iff $a \sqsubseteq b$.

(43) FixPoints(f) is complete.

Let us consider L, f. The functor lfp(f) yielding an element of L is defined by:

(Def. 12) $\mathrm{lfp}(f) = f^{(\text{the carrier of } L)^{+}}_{\sqcup}(\perp_L)$.

The functor gfp(f) yielding an element of L is defined by:

(Def. 13) $\mathrm{gfp}(f) = f^{(\text{the carrier of } L)^{+}}_{\sqcap}(\top_L)$.

One can prove the following propositions:

(44) lfp(f) is a fixpoint of f and there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^{O}_{\sqcup}(\perp_L) = \mathrm{lfp}(f)$.

(45) gfp(f) is a fixpoint of f and there exists O such that $\overline{\overline{O}} \leq \overline{\overline{\text{the carrier of } L}}$ and $f^{O}_{\sqcap}(\top_L) = \mathrm{gfp}(f)$.

(46) If a is a fixpoint of f, then $\mathrm{lfp}(f) \sqsubseteq a$ and $a \sqsubseteq \mathrm{gfp}(f)$.

(47) If $f(a) \sqsubseteq a$, then $\mathrm{lfp}(f) \sqsubseteq a$.

(48) If $a \sqsubseteq f(a)$, then $a \sqsubseteq \mathrm{gfp}(f)$.

5. Boolean Lattices

In the sequel f denotes a monotone unary operation on the lattice of subsets of A.

Let A be a set. Observe that the lattice of subsets of A is complete.

Next we state three propositions:

(49) Let f be a unary operation on the lattice of subsets of A. Then f is monotone if and only if f is $\subseteq$-monotone.

(50) There exists a $\subseteq$-monotone function g from 2^A into 2^A such that $\mathrm{lfp}(A, g) = \mathrm{lfp}(f)$.

(51) There exists a $\subseteq$-monotone function g from 2^A into 2^A such that $\mathrm{gfp}(A, g) = \mathrm{gfp}(f)$.

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