JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

A Decomposition of Simple Closed Curves and the Order of Their Points

Yatsuka Nakamura
Shinshu University
Nagano

Andrzej Trybulec
University of Białystok

Summary. The goal of the article is to introduce an order on a simple closed curve. To do this, we fix two points on the curve and devide it into two arcs. We prove that such a decomposition is unique. Other auxiliary theorems about arcs are proven for preparation of the proof of the above.

MML Identifier: `JORDAN6`.

WWW: `http://mizar.org/JFM/Vol9/jordan6.html`

The articles [18], [21], [1], [20], [12], [17], [22], [3], [4], [9], [10], [15], [8], [16], [6], [19], [7], [13], [2], [14], [11], and [5] provide the notation and terminology for this paper.

1. Middle Points of Arcs

We follow the rules: s, r denote real numbers, n denotes a natural number, and p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

We now state a number of propositions:

(2)[1] If $r \leq s$, then $r \leq \frac{r+s}{2}$ and $\frac{r+s}{2} \leq s$.

(3) Let T_1 be a non empty topological space, P be a subset of T_1, A be a subset of $T_1 \restriction P$, and B be a subset of T_1. If B is closed and $A = B \cap P$, then A is closed.

(4) Let T_1, T_2 be non empty topological spaces, P be a non empty subset of T_2, and f be a map from T_1 into $T_2 \restriction P$. Then

(i) f is a map from T_1 into T_2, and

(ii) for every map f_2 from T_1 into T_2 such that $f_2 = f$ and f is continuous holds f_2 is continuous.

(5) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} \geq r\}$ holds P is closed.

(6) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} \leq r\}$ holds P is closed.

(7) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = r\}$ holds P is closed.

[1] The proposition (1) has been removed.

(8) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p;\, p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \geq r\}$ holds P is closed.

(9) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p;\, p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \leq r\}$ holds P is closed.

(10) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p;\, p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} = r\}$ holds P is closed.

(11) For every subset P of $\mathcal{E}^n_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p_1 to p_2 holds P is connected.

(12) For every subset P of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that P is an arc from p_1 to p_2 holds P is closed.

(13) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in P$ and $q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}$.

(14) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

(15) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{xMiddle}(P, p_1, p_2)$ yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) For every subset Q of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}\}$ holds $\mathrm{xMiddle}(P, p_1, p_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$.

Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{yMiddle}(P, p_1, p_2)$ yielding a point of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) For every subset Q of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}}{2}\}$ holds $\mathrm{yMiddle}(P, p_1, p_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$.

One can prove the following three propositions:

(16) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $\mathrm{xMiddle}(P, p_1, p_2) \in P$ and $\mathrm{yMiddle}(P, p_1, p_2) \in P$.

(17) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 = \mathrm{xMiddle}(P, p_1, p_2)$ iff $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$.

(18) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 = \mathrm{yMiddle}(P, p_1, p_2)$ iff $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$.

2. Segments of Arcs

The following proposition is true

(19) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2, then LE q_2, q_1, P, p_2, p_1.

Let P be a subset of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{LSegment}(P, p_1, p_2, q_1)$ yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 3) $\mathrm{LSegment}(P, p_1, p_2, q_1) = \{q : \mathrm{LE}\ q,\, q_1,\, P,\, p_1,\, p_2\}$.

Let P be a subset of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{RSegment}(P, p_1, p_2, q_1)$ yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 4) $\mathrm{RSegment}(P, p_1, p_2, q_1) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2\}$.

The following propositions are true:

(20) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LSegment}(P, p_1, p_2, q_1) \subseteq P$.

(21) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{RSegment}(P, p_1, p_2, q_1) \subseteq P$.

(22) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{LSegment}(P, p_1, p_2, p_1) = \{p_1\}$.

(25)[2] For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{LSegment}(P, p_1, p_2, p_2) = P$.

(26) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{RSegment}(P, p_1, p_2, p_2) = \{p_2\}$.

(27) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{RSegment}(P, p_1, p_2, p_1) = P$.

(28) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then $\mathrm{RSegment}(P, p_1, p_2, q_1) = \mathrm{LSegment}(P, p_2, p_1, q_1)$.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 5) $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{RSegment}(P, p_1, p_2, q_1) \cap \mathrm{LSegment}(P, p_1, p_2, q_2)$.

One can prove the following four propositions:

(29) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2 \wedge \mathrm{LE}\ q, q_2, P, p_1, p_2\}$.

(30) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then LE q_1, q_2, P, p_1, p_2 if and only if LE q_2, q_1, P, p_2, p_1.

(31) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \in P$, then $\mathrm{LSegment}(P, p_1, p_2, q) = \mathrm{RSegment}(P, p_2, p_1, q)$.

(32) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$, then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{Segment}(P, p_2, p_1, q_2, q_1)$.

3. Decomposition of a Simple Closed Curve Into Two Arcs

Let s be a real number. The functor $\mathrm{VerticalLine}(s)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 6) $\mathrm{VerticalLine}(s) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = s\}$.

The functor $\mathrm{HorizontalLine}(s)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 7) $\mathrm{HorizontalLine}(s) = \{p : p_{\mathbf{2}} = s\}$.

We now state four propositions:

(33) For every real number r holds $\mathrm{VerticalLine}(r)$ is closed and $\mathrm{HorizontalLine}(r)$ is closed.

(34) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p \in \mathrm{VerticalLine}(r)$ iff $p_{\mathbf{1}} = r$.

(35) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p \in \mathrm{HorizontalLine}(r)$ iff $p_{\mathbf{2}} = r$.

[2] The propositions (23) and (24) have been removed.

(40)[3] Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve. Then there exist non empty subsets P_1, P_2 of $\mathcal{E}^2_{\rm T}$ such that

(i) P_1 is an arc from $\mathrm{W}_{\min}(P)$ to $\mathrm{E}_{\max}(P)$,

(ii) P_2 is an arc from $\mathrm{E}_{\max}(P)$ to $\mathrm{W}_{\min}(P)$,

(iii) $P_1 \cap P_2 = \{\mathrm{W}_{\min}(P), \mathrm{E}_{\max}(P)\}$,

(iv) $P_1 \cup P_2 = P$, and

(v) $(\mathrm{FPoint}(P_1, \mathrm{W}_{\min}(P), \mathrm{E}_{\max}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \mathrm{E}_{\max}(P), \mathrm{W}_{\min}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

4. Uniqueness of Decomposition of a Simple Closed Curve

The following propositions are true:

(41) For every subset P of $\mathbb{I}$ such that $P =$ (the carrier of $\mathbb{I}$) $\setminus \{0, 1\}$ holds P is open.

(44)[4] For all real numbers r, s holds $]r, s[$ misses $\{r, s\}$.

(45) For all real numbers a, b, c holds $c \in]a, b[$ iff $a < c$ and $c < b$.

(46) For every subset P of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers r, s such that $P =]r, s[$ holds P is open.

(47) Let S be a non empty topological space, P_1, P_2 be subsets of S, and P'_1 be a subset of $S \restriction P_2$. If $P_1 = P'_1$ and $P_1 \subseteq P_2$, then $S \restriction P_1 = S \restriction P_2 \restriction P'_1$.

(48) For every subset P_7 of $\mathbb{I}$ such that $P_7 =$ (the carrier of $\mathbb{I}$) $\setminus \{0, 1\}$ holds $P_7 \neq \emptyset$ and P_7 is connected.

(49) For every subset P of $\mathcal{E}^n_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p_1 to p_2 holds $p_1 \neq p_2$.

(50) Let P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is open.

(52)[5] Let P be a subset of $\mathcal{E}^n_{\rm T}$, P_1, P_2 be non empty subsets of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose $p_1 \in P$ and $p_2 \in P$ and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \{p_1, p_2\}$ and $Q = P_1 \setminus \{p_1, p_2\}$. Then Q is open.

(53) Let P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is connected.

(54) Let G_1 be a non empty topological space, P_1, P be subsets of G_1, Q' be a subset of $G_1 \restriction P_1$, and Q be a subset of $G_1 \restriction P$. If $P_1 \subseteq P$ and $Q = Q'$ and Q' is connected, then Q is connected.

(55) Let P be a subset of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point p_3 of $\mathcal{E}^n_{\rm T}$ such that $p_3 \in P$ and $p_3 \neq p_1$ and $p_3 \neq p_2$.

(56) For every subset P of $\mathcal{E}^n_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p_1 to p_2 holds $P \setminus \{p_1, p_2\} \neq \emptyset$.

(57) Let P_1 be a subset of $\mathcal{E}^n_{\rm T}$, P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P_1 is an arc from p_1 to p_2 and $P_1 \subseteq P$ and $Q = P_1 \setminus \{p_1, p_2\}$, then Q is connected.

(58) Let T, S, V be non empty topological spaces, P_1 be a non empty subset of S, P_2 be a subset of S, f be a map from T into $S \restriction P_1$, and g be a map from $S \restriction P_2$ into V. Suppose $P_1 \subseteq P_2$ and f is continuous and g is continuous. Then there exists a map h from T into V such that $h = g \cdot f$ and h is continuous.

[3] The propositions (36)–(39) have been removed.
[4] The propositions (42) and (43) have been removed.
[5] The proposition (51) has been removed.

(59) Let P_1, P_2 be subsets of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(60) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, Q be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_1 \neq p_2$ and $Q = P \setminus \{p_1, p_2\}$. Then Q is not connected.

(61) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, P_1, P_2, P'_1, P'_2 be subsets of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that P is a simple closed curve and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and P'_1 is an arc from p_1 to p_2 and P'_2 is an arc from p_1 to p_2 and $P'_1 \cup P'_2 = P$. Then $P_1 = P'_1$ and $P_2 = P'_2$ or $P_1 = P'_2$ and $P_2 = P'_1$.

5. Lower Arcs and Upper Arcs

Let us observe that every element of $\mathbb{R}^{\mathbf{1}}$ is real.

One can prove the following proposition

(64)[6] Let P_1 be a subset of $\mathcal{E}^2_{\rm T}$, r be a real number, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $(p_1)_{\mathbf{1}} \leq r$ and $r \leq (p_2)_{\mathbf{1}}$ and P_1 is an arc from p_1 to p_2. Then P_1 meets VerticalLine(r) and $P_1 \cap$ VerticalLine(r) is closed.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Let us assume that P is a simple closed curve. The functor UpperArc(P) yielding a non empty subset of $\mathcal{E}^2_{\rm T}$ is defined by the conditions (Def. 8).

(Def. 8)(i) UpperArc(P) is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$, and

(ii) there exists a non empty subset P_2 of $\mathcal{E}^2_{\rm T}$ such that P_2 is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{UpperArc}(P) \cap P_2 = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup P_2 = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Let us assume that P is a simple closed curve. The functor LowerArc(P) yielding a non empty subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 9) LowerArc(P) is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{UpperArc}(P) \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup \mathrm{LowerArc}(P) = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(\mathrm{LowerArc}(P), \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

The following propositions are true:

(65) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve. Then UpperArc(P) is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$ and UpperArc(P) is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and LowerArc(P) is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and LowerArc(P) is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$ and $\mathrm{UpperArc}(P) \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup \mathrm{LowerArc}(P) = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(\mathrm{LowerArc}(P), \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

(66) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve, then $\mathrm{LowerArc}(P) = (P \setminus \mathrm{UpperArc}(P)) \cup \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) = (P \setminus \mathrm{LowerArc}(P)) \cup \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$.

(67) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and P_1 be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$. If P is a simple closed curve and $\mathrm{UpperArc}(P) \cap P_1 = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup P_1 = P$, then $P_1 = \mathrm{LowerArc}(P)$.

(68) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and P_1 be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$. If P is a simple closed curve and $P_1 \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $P_1 \cup \mathrm{LowerArc}(P) = P$, then $P_1 = \mathrm{UpperArc}(P)$.

[6] The propositions (62) and (63) have been removed.

6. An Order of Points in a Simple Closed Curve

We now state two propositions:

(69) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and LE q, p_1, P, p_1, p_2, then $q = p_1$.

(70) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and LE p_2, q, P, p_1, p_2, then $q = p_2$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate $q_1 \leq_P q_2$ is defined by the conditions (Def. 10).

(Def. 10)(i) $q_1 \in \mathrm{UpperArc}(P)$ and $q_2 \in \mathrm{LowerArc}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$, or

(ii) $q_1 \in \mathrm{UpperArc}(P)$ and $q_2 \in \mathrm{UpperArc}(P)$ and LE q_1, q_2, $\mathrm{UpperArc}(P)$, $\mathrm{W_{min}}(P)$, $\mathrm{E_{max}}(P)$, or

(iii) $q_1 \in \mathrm{LowerArc}(P)$ and $q_2 \in \mathrm{LowerArc}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$ and LE q_1, q_2, $\mathrm{LowerArc}(P)$, $\mathrm{E_{max}}(P)$, $\mathrm{W_{min}}(P)$.

One can prove the following propositions:

(71) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $q \in P$, then $q \leq_P q$.

(72) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_1$, then $q_1 = q_2$.

(73) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2, q_3 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$, then $q_1 \leq_P q_3$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/pscomp_1.html.

[6] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[7] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[8] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topmetr.html.

[9] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal2.html.

[11] Adam Grabowski and Yatsuka Nakamura. The ordering of points on a curve. Part II. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/jordan5c.html.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/metric_1.html.

[14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[15] Beata Padlewska. Connected spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/connsp_1.html`.

[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[17] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[19] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/borsuk_1.html`.

[20] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received December 19, 1997

Published January 2, 2004