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Bounding Boxes for Special Sequences in $\mathcal{E}^2$

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Summary. This is the continuation of the proof of the Jordan Theorem according to [18].

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The articles [19], [23], [2], [21], [20], [1], [16], [24], [3], [4], [22], [6], [11], [10], [9], [8], [12], [13], [15], [17], [7], [14], and [5] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we use the following convention: p, q are points of $\mathcal{E}^2_{\rm T}$, r is a real number, h is a non constant standard special circular sequence, g is a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, f is a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and I, i_1, i, j, k are natural numbers.

The following propositions are true:

(3)[1] For every natural number n and for every finite sequence h of elements of $\mathcal{E}^n_{\rm T}$ such that $\operatorname{len} h \geq 2$ holds $h_{\operatorname{len} h} \in \mathcal{L}(h, \operatorname{len} h -' 1)$.

(4) If $3 \leq i$, then $i \bmod (i -' 1) = 1$.

(5) If $p \in \operatorname{rng} h$, then there exists a natural number i such that $1 \leq i$ and $i+1 \leq \operatorname{len} h$ and $h(i) = p$.

(6) For every finite sequence g of elements of $\mathbb{R}$ such that $r \in \operatorname{rng} g$ holds $(\operatorname{Inc}(g))(1) \leq r$ and $r \leq (\operatorname{Inc}(g))(\operatorname{len} \operatorname{Inc}(g))$.

(7) Suppose $1 \leq i$ and $i \leq \operatorname{len} h$ and $1 \leq I$ and $I \leq \operatorname{width}$ the Go-board of h. Then $(\text{the Go-board of } h \circ (1, I))_{\mathbf{1}} \leq (h_i)_{\mathbf{1}}$ and $(h_i)_{\mathbf{1}} \leq (\text{the Go-board of } h \circ (\operatorname{len} \text{the Go-board of } h, I))_{\mathbf{1}}$.

(8) Suppose $1 \leq i$ and $i \leq \operatorname{len} h$ and $1 \leq I$ and $I \leq \operatorname{len}$ the Go-board of h. Then $(\text{the Go-board of } h \circ (I, 1))_{\mathbf{2}} \leq (h_i)_{\mathbf{2}}$ and $(h_i)_{\mathbf{2}} \leq (\text{the Go-board of } h \circ (I, \operatorname{width} \text{the Go-board of } h))_{\mathbf{2}}$.

(9) Suppose $1 \leq i$ and $i \leq \operatorname{len}$ the Go-board of f. Then there exist k, j such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $f_k =$ the Go-board of $f \circ (i, j)$.

(10) Suppose $1 \leq j$ and $j \leq \operatorname{width}$ the Go-board of f. Then there exist k, i such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $f_k =$ the Go-board of $f \circ (i, j)$.

[1] The propositions (1) and (2) have been removed.

(11) Suppose $1 \leq i$ and $i \leq \mathrm{len}$ the Go-board of f and $1 \leq j$ and $j \leq \mathrm{width}$ the Go-board of f. Then there exists k such that $k \in \mathrm{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $(f_k)_{\mathbf{1}} =$ (the Go-board of $f \circ (i, j))_{\mathbf{1}}$.

(12) Suppose $1 \leq i$ and $i \leq \mathrm{len}$ the Go-board of f and $1 \leq j$ and $j \leq \mathrm{width}$ the Go-board of f. Then there exists k such that $k \in \mathrm{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $(f_k)_{\mathbf{2}} =$ (the Go-board of $f \circ (i, j))_{\mathbf{2}}$.

2. Extrema of Projections

We now state a number of propositions:

(13) If $1 \leq i$ and $i \leq \mathrm{len}\, h$, then S-bound$(\widetilde{\mathcal{L}}(h)) \leq (h_i)_{\mathbf{2}}$ and $(h_i)_{\mathbf{2}} \leq$ N-bound$(\widetilde{\mathcal{L}}(h))$.

(14) If $1 \leq i$ and $i \leq \mathrm{len}\, h$, then W-bound$(\widetilde{\mathcal{L}}(h)) \leq (h_i)_{\mathbf{1}}$ and $(h_i)_{\mathbf{1}} \leq$ E-bound$(\widetilde{\mathcal{L}}(h))$.

(15) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\mathrm{proj2} \restriction \mathrm{W_{most}}(\widetilde{\mathcal{L}}(h)))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{W_{most}}(\widetilde{\mathcal{L}}(h))$).

(16) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\mathrm{proj2} \restriction \mathrm{E_{most}}(\widetilde{\mathcal{L}}(h)))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{E_{most}}(\widetilde{\mathcal{L}}(h))$).

(17) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\mathrm{proj1} \restriction \mathrm{N_{most}}(\widetilde{\mathcal{L}}(h)))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{N_{most}}(\widetilde{\mathcal{L}}(h))$).

(18) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\mathrm{proj1} \restriction \mathrm{S_{most}}(\widetilde{\mathcal{L}}(h)))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{S_{most}}(\widetilde{\mathcal{L}}(h))$).

(19) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $X = (\mathrm{proj1} \restriction \widetilde{\mathcal{L}}(g))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \widetilde{\mathcal{L}}(g)$).

(20) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $X = (\mathrm{proj2} \restriction \widetilde{\mathcal{L}}(g))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \widetilde{\mathcal{L}}(g)$).

(21) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\mathrm{proj2} \restriction \mathrm{W_{most}}(\widetilde{\mathcal{L}}(h)))$.

(22) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\mathrm{proj2} \restriction \mathrm{W_{most}}(\widetilde{\mathcal{L}}(h)))$.

(23) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\mathrm{proj2} \restriction \mathrm{E_{most}}(\widetilde{\mathcal{L}}(h)))$.

(24) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\mathrm{proj2} \restriction \mathrm{E_{most}}(\widetilde{\mathcal{L}}(h)))$.

(25) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\inf X = \inf(\mathrm{proj1} \restriction \widetilde{\mathcal{L}}(g))$.

(26) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\mathrm{proj1} \restriction \mathrm{S_{most}}(\widetilde{\mathcal{L}}(h)))$.

(27) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\mathrm{proj1} \restriction \mathrm{S_{most}}(\widetilde{\mathcal{L}}(h)))$.

(28) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\mathrm{proj1} \restriction \mathrm{N_{most}}(\widetilde{\mathcal{L}}(h)))$.

(29) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(h)) \ \wedge\ q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\mathrm{proj1} \restriction \mathrm{N_{most}}(\widetilde{\mathcal{L}}(h)))$.

(30) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\inf X = \inf(\mathrm{proj2} \restriction \widetilde{\mathcal{L}}(g))$.

(31) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\sup X = \sup(\mathrm{proj1} \restriction \widetilde{\mathcal{L}}(g))$.

(32) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\sup X = \sup(\mathrm{proj2} \restriction \widetilde{\mathcal{L}}(g))$.

(33) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leq I$ and $I \leq$ width the Go-board of h, then (the Go-board of $h \circ (1, I))_{\mathbf{1}} \leq p_{\mathbf{1}}$.

(34) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leq I$ and $I \leq$ width the Go-board of h, then $p_{\mathbf{1}} \leq$ (the Go-board of $h \circ$ (len the Go-board of h, I))$_{\mathbf{1}}$.

(35) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leq I$ and $I \leq$ len the Go-board of h, then (the Go-board of $h \circ (I, 1))_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(36) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leq I$ and $I \leq$ len the Go-board of h, then $p_{\mathbf{2}} \leq$ (the Go-board of $h \circ$ (I, width the Go-board of h))$_{\mathbf{2}}$.

(37) Suppose $1 \leq i$ and $i \leq$ len the Go-board of h and $1 \leq j$ and $j \leq$ width the Go-board of h. Then there exists q such that $q_{\mathbf{1}} =$ (the Go-board of $h \circ (i, j))_{\mathbf{1}}$ and $q \in \widetilde{\mathcal{L}}(h)$.

(38) Suppose $1 \leq i$ and $i \leq$ len the Go-board of h and $1 \leq j$ and $j \leq$ width the Go-board of h. Then there exists q such that $q_{\mathbf{2}} =$ (the Go-board of $h \circ (i, j))_{\mathbf{2}}$ and $q \in \widetilde{\mathcal{L}}(h)$.

(39) W-bound$(\widetilde{\mathcal{L}}(h)) =$ (the Go-board of $h \circ (1, 1))_{\mathbf{1}}$.

(40) S-bound$(\widetilde{\mathcal{L}}(h)) =$ (the Go-board of $h \circ (1, 1))_{\mathbf{2}}$.

(41) E-bound$(\widetilde{\mathcal{L}}(h)) =$ (the Go-board of $h \circ$ (len the Go-board of h, 1))$_{\mathbf{1}}$.

(42) N-bound$(\widetilde{\mathcal{L}}(h)) =$ (the Go-board of $h \circ$ (1, width the Go-board of h))$_{\mathbf{2}}$.

(43) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that $1 \leq i$ and $i \leq \mathrm{len}\, f$ and $1 \leq I$ and $I \leq$ len the Go-board of f and $Y = \{j : \langle I, j\rangle \in$ the indices of the Go-board of $f \wedge \bigvee_k (k \in \mathrm{dom}\, f \wedge f_k =$ the Go-board of $f \circ (I, j))\}$ and $(f_i)_{\mathbf{1}} =$ (the Go-board of $f \circ (I, 1))_{\mathbf{1}}$ and $i_1 = \min Y$. Then (the Go-board of $f \circ (I, i_1))_{\mathbf{2}} \leq (f_i)_{\mathbf{2}}$.

(44) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that $1 \leq i$ and $i \leq \mathrm{len}\, h$ and $1 \leq I$ and $I \leq$ width the Go-board of h and $Y = \{j : \langle j, I\rangle \in$ the indices of the Go-board of $h \wedge \bigvee_k (k \in \mathrm{dom}\, h \wedge h_k =$ the Go-board of $h \circ (j, I))\}$ and $(h_i)_{\mathbf{2}} =$ (the Go-board of $h \circ (1, I))_{\mathbf{2}}$ and $i_1 = \min Y$. Then (the Go-board of $h \circ (i_1, I))_{\mathbf{1}} \leq (h_i)_{\mathbf{1}}$.

(45) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that $1 \leq i$ and $i \leq \mathrm{len}\, h$ and $1 \leq I$ and $I \leq$ width the Go-board of h and $Y = \{j : \langle j, I\rangle \in$ the indices of the Go-board of $h \wedge \bigvee_k (k \in \mathrm{dom}\, h \wedge h_k =$ the Go-board of $h \circ (j, I))\}$ and $(h_i)_{\mathbf{2}} =$ (the Go-board of $h \circ (1, I))_{\mathbf{2}}$ and $i_1 = \max Y$. Then (the Go-board of $h \circ (i_1, I))_{\mathbf{1}} \geq (h_i)_{\mathbf{1}}$.

(46) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that $1 \leq i$ and $i \leq \mathrm{len}\, f$ and $1 \leq I$ and $I \leq$ len the Go-board of f and $Y = \{j : \langle I, j\rangle \in$ the indices of the Go-board of $f \wedge \bigvee_k (k \in \mathrm{dom}\, f \wedge f_k =$ the Go-board of $f \circ (I, j))\}$ and $(f_i)_{\mathbf{1}} =$ (the Go-board of $f \circ (I, 1))_{\mathbf{1}}$ and $i_1 = \max Y$. Then (the Go-board of $f \circ (I, i_1))_{\mathbf{2}} \geq (f_i)_{\mathbf{2}}$.

3. Coordinates of the Special Circular Sequences Bounding Boxes

Let g be a non constant standard special circular sequence. The functor $\mathrm{i}_{\mathrm{SW}}\, g$ yielding a natural number is defined by:

(Def. 1) $\langle 1, \mathrm{i}_{\mathrm{SW}}\, g\rangle \in$ the indices of the Go-board of g and the Go-board of $g \circ (1, \mathrm{i}_{\mathrm{SW}}\, g) = \mathrm{W}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{NW}}\, g$ yields a natural number and is defined as follows:

(Def. 2) $\langle 1, \mathrm{i}_{\mathrm{NW}}\, g\rangle \in$ the indices of the Go-board of g and the Go-board of $g \circ (1, \mathrm{i}_{\mathrm{NW}}\, g) = \mathrm{W}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{SE}}\, g$ yields a natural number and is defined by the conditions (Def. 3).

(Def. 3)(i) $\langle \mathrm{len}\,$the Go-board of g, $\mathrm{i}_{\mathrm{SE}}\, g\rangle \in$ the indices of the Go-board of g, and

(ii) the Go-board of $g \circ (\mathrm{len}\,$the Go-board of g, $\mathrm{i}_{\mathrm{SE}}\, g) = \mathrm{E}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{NE}}\, g$ yields a natural number and is defined by the conditions (Def. 4).

(Def. 4)(i) $\langle \mathrm{len}\,$the Go-board of g, $\mathrm{i}_{\mathrm{NE}}\, g\rangle \in$ the indices of the Go-board of g, and

(ii) the Go-board of $g \circ (\mathrm{len}\,$the Go-board of g, $\mathrm{i}_{\mathrm{NE}}\, g) = \mathrm{E}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{WS}}\, g$ yields a natural number and is defined by:

(Def. 5) $\langle \mathrm{i}_{\mathrm{WS}}\, g, 1\rangle \in$ the indices of the Go-board of g and the Go-board of $g \circ (\mathrm{i}_{\mathrm{WS}}\, g, 1) = \mathrm{S}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{ES}}\, g$ yields a natural number and is defined by:

(Def. 6) $\langle \mathrm{i}_{\mathrm{ES}}\, g, 1\rangle \in$ the indices of the Go-board of g and the Go-board of $g \circ (\mathrm{i}_{\mathrm{ES}}\, g, 1) = \mathrm{S}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{WN}}\, g$ yields a natural number and is defined by the conditions (Def. 7).

(Def. 7)(i) $\langle \mathrm{i}_{\mathrm{WN}}\, g$, $\mathrm{width}\,$the Go-board of $g\rangle \in$ the indices of the Go-board of g, and

(ii) the Go-board of $g \circ (\mathrm{i}_{\mathrm{WN}}\, g, \mathrm{width}\,$the Go-board of $g) = \mathrm{N}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{i}_{\mathrm{EN}}\, g$ yields a natural number and is defined by the conditions (Def. 8).

(Def. 8)(i) $\langle \mathrm{i}_{\mathrm{EN}}\, g$, $\mathrm{width}\,$the Go-board of $g\rangle \in$ the indices of the Go-board of g, and

(ii) the Go-board of $g \circ (\mathrm{i}_{\mathrm{EN}}\, g, \mathrm{width}\,$the Go-board of $g) = \mathrm{N}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

One can prove the following propositions:

(47) $1 \leq \mathrm{i}_{\mathrm{WN}}\, h$ and $\mathrm{i}_{\mathrm{WN}}\, h \leq \mathrm{len}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{EN}}\, h$ and $\mathrm{i}_{\mathrm{EN}}\, h \leq \mathrm{len}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{WS}}\, h$ and $\mathrm{i}_{\mathrm{WS}}\, h \leq \mathrm{len}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{ES}}\, h$ and $\mathrm{i}_{\mathrm{ES}}\, h \leq \mathrm{len}\,$the Go-board of h.

(48) $1 \leq \mathrm{i}_{\mathrm{NE}}\, h$ and $\mathrm{i}_{\mathrm{NE}}\, h \leq \mathrm{width}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{SE}}\, h$ and $\mathrm{i}_{\mathrm{SE}}\, h \leq \mathrm{width}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{NW}}\, h$ and $\mathrm{i}_{\mathrm{NW}}\, h \leq \mathrm{width}\,$the Go-board of h and $1 \leq \mathrm{i}_{\mathrm{SW}}\, h$ and $\mathrm{i}_{\mathrm{SW}}\, h \leq \mathrm{width}\,$the Go-board of h.

Let g be a non constant standard special circular sequence. The functor $\mathrm{n}_{\mathrm{SW}}\, g$ yielding a natural number is defined by:

(Def. 9) $1 \leq \mathrm{n}_{\mathrm{SW}}\, g$ and $\mathrm{n}_{\mathrm{SW}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n}_{\mathrm{SW}}\, g) = \mathrm{W}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n}_{\mathrm{NW}}\, g$ yielding a natural number is defined by:

(Def. 10) $1 \leq \mathrm{n}_{\mathrm{NW}}\, g$ and $\mathrm{n}_{\mathrm{NW}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n}_{\mathrm{NW}}\, g) = \mathrm{W}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n}_{\mathrm{SE}}\, g$ yields a natural number and is defined as follows:

(Def. 11) $1 \leq \mathrm{n}_{\mathrm{SE}}\, g$ and $\mathrm{n}_{\mathrm{SE}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n}_{\mathrm{SE}}\, g) = \mathrm{E}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n}_{\mathrm{NE}}\, g$ yielding a natural number is defined as follows:

(Def. 12) $1 \leq \mathrm{n}_{\mathrm{NE}}\, g$ and $\mathrm{n}_{\mathrm{NE}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n}_{\mathrm{NE}}\, g) = \mathrm{E}_{\mathrm{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n}_{\mathrm{WS}}\, g$ yields a natural number and is defined as follows:

(Def. 13) $1 \leq \mathrm{n}_{\mathrm{WS}}\, g$ and $\mathrm{n}_{\mathrm{WS}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n}_{\mathrm{WS}}\, g) = \mathrm{S}_{\mathrm{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n}_{\mathrm{ES}}\, g$ yields a natural number and is defined as follows:

(Def. 14) $1 \leq \mathrm{n_{ES}}\, g$ and $\mathrm{n_{ES}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n_{ES}}\, g) = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n_{WN}}\, g$ yielding a natural number is defined by:

(Def. 15) $1 \leq \mathrm{n_{WN}}\, g$ and $\mathrm{n_{WN}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n_{WN}}\, g) = \mathrm{N_{min}}(\widetilde{\mathcal{L}}(g))$.

The functor $\mathrm{n_{EN}}\, g$ yielding a natural number is defined by:

(Def. 16) $1 \leq \mathrm{n_{EN}}\, g$ and $\mathrm{n_{EN}}\, g + 1 \leq \mathrm{len}\, g$ and $g(\mathrm{n_{EN}}\, g) = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(g))$.

The following four propositions are true:

(49) $\mathrm{n_{WN}}\, h \neq \mathrm{n_{WS}}\, h$.

(50) $\mathrm{n_{SW}}\, h \neq \mathrm{n_{SE}}\, h$.

(51) $\mathrm{n_{EN}}\, h \neq \mathrm{n_{ES}}\, h$.

(52) $\mathrm{n_{NW}}\, h \neq \mathrm{n_{NE}}\, h$.

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