JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Ordering of Points on a Curve. Part I

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Summary. Some auxiliary theorems needed to formalize the proof of the Jordan Curve Theorem according to [20] are proved.

MML Identifier: JORDAN5B.

WWW: http://mizar.org/JFM/Vol9/jordan5b.html

The articles [21], [9], [1], [17], [19], [24], [2], [4], [5], [23], [15], [8], [18], [6], [22], [7], [11], [12], [13], [10], [14], [16], and [3] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For every natural number i_1 such that $1 \leq i_1$ holds $i_1 -' 1 < i_1$.

(2) For all natural numbers i, k such that $i+1 \leq k$ holds $1 \leq k -' i$.

(3) For all natural numbers i, k such that $1 \leq i$ and $1 \leq k$ holds $(k -' i) + 1 \leq k$.

(4) For every real number r such that $r \in$ the carrier of $\mathbb{I}$ holds $1 - r \in$ the carrier of $\mathbb{I}$.

(5) For all points p, q, p_1 of $\mathcal{E}^2_\mathrm{T}$ such that $p_\mathbf{2} \neq q_\mathbf{2}$ and $p_1 \in \mathcal{L}(p,q)$ holds if $(p_1)_\mathbf{2} = p_\mathbf{2}$, then $p_1 = p$.

(6) For all points p, q, p_1 of $\mathcal{E}^2_\mathrm{T}$ such that $p_\mathbf{1} \neq q_\mathbf{1}$ and $p_1 \in \mathcal{L}(p,q)$ holds if $(p_1)_\mathbf{1} = p_\mathbf{1}$, then $p_1 = p$.

(7) Let f be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, P be a non empty subset of $\mathcal{E}^2_\mathrm{T}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_\mathrm{T})\restriction P$, and i be a natural number. Suppose $1 \leq i$ and $i+1 \leq \mathrm{len} f$ and f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and F is a homeomorphism and $F(0) = f_1$ and $F(1) = f_{\mathrm{len} f}$. Then there exist real numbers p_1, p_2 such that $p_1 < p_2$ and $0 \leq p_1$ and $p_1 \leq 1$ and $0 \leq p_2$ and $p_2 \leq 1$ and $\mathcal{L}(f,i) = F^\circ[p_1, p_2]$ and $F(p_1) = f_i$ and $F(p_2) = f_{i+1}$.

(8) Let f be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, Q, R be non empty subsets of $\mathcal{E}^2_\mathrm{T}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_\mathrm{T})\restriction Q$, i be a natural number, and P be a non empty subset of $\mathbb{I}$. Suppose that f is a special sequence and F is a homeomorphism and $F(0) = f_1$ and $F(1) = f_{\mathrm{len} f}$ and $1 \leq i$ and $i+1 \leq \mathrm{len} f$ and $F^\circ P = \mathcal{L}(f,i)$ and $Q = \widetilde{\mathcal{L}}(f)$ and $R = \mathcal{L}(f,i)$. Then there exists a map G from $\mathbb{I}\restriction P$ into $(\mathcal{E}^2_\mathrm{T})\restriction R$ such that $G = F\restriction P$ and G is a homeomorphism.

2. Some properties of real intervals

We now state several propositions:

(9) For all points p_1, p_2, p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $p \leq_{p_1,p_2} p$.

(10) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $p_1 \leq_{p_1,p_2} p$.

(11) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $p \leq_{p_1,p_2} p_2$.

(12) For all points p_1, p_2, q_1, q_2, q_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $q_1 \leq_{p_1,p_2} q_2$ and $q_2 \leq_{p_1,p_2} q_3$ holds $q_1 \leq_{p_1,p_2} q_3$.

(13) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $\mathcal{L}(p,q) = \{p_1; p_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p \leq_{p,q} p_1 \wedge p_1 \leq_{p,q} q\}$.

(14) Let n be a natural number, P be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then P is an arc from p_2 to p_1.

(15) Let i be a natural number, f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $P = \mathcal{L}(f,i)$. Then P is an arc from f_i to f_{i+1}.

3. Cutting off sequences

The following propositions are true:

(16) Let g_1 be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose $1 \leq i$ and $i \leq \mathrm{len}\, g_1$ and g_1 is a special sequence. If $(g_1)_1 \in \widetilde{\mathcal{L}}(\mathrm{mid}(g_1, i, \mathrm{len}\, g_1))$, then $i = 1$.

(17) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p = f(\mathrm{len}\, f)$, then $\downharpoonright p, f = \langle p \rangle$.

(21)[1] Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and f is a special sequence, then $\mathrm{Index}(p, \downharpoonright p, f) = 1$.

(22) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(\mathrm{len}\, f)$, then $p \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

(23) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(1)$, then $p \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$.

(24) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $\downharpoonright\downharpoonleft p, f, p = \langle p \rangle$.

(25) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $q \neq f(\mathrm{len}\, f)$ and $p = f(\mathrm{len}\, f)$ and f is a special sequence. Then $p \in \widetilde{\mathcal{L}}(\downharpoonright q, f)$.

(26) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \neq f(\mathrm{len}\, f)$ or $q \neq f(\mathrm{len}\, f)$ and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence. Then $p \in \widetilde{\mathcal{L}}(\downharpoonright q, f)$ or $q \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

(27) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonright\downharpoonleft p, f, q) \subseteq \widetilde{\mathcal{L}}(f)$.

[1] The propositions (18)–(20) have been removed.

(28) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leq i$ and $j \leq \mathrm{len}\,\text{the Go-board of } f$ and $i < j$. Then $\mathcal{L}(\text{the Go-board of } f \circ (1, \mathrm{width}\,\text{the Go-board of } f), \text{the Go-board of } f \circ (i, \mathrm{width}\,\text{the Go-board of } f)) \cap \mathcal{L}(\text{the Go-board of } f \circ (j, \mathrm{width}\,\text{the Go-board of } f), \text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f, \mathrm{width}\,\text{the Go-board of } f)) = \emptyset$.

(29) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leq i$ and $j \leq \mathrm{width}\,\text{the Go-board of } f$ and $i < j$. Then $\mathcal{L}(\text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f,\ 1), \text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f,\ i)) \cap \mathcal{L}(\text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f,\ j), \text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f, \mathrm{width}\,\text{the Go-board of } f)) = \emptyset$.

(30) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence, then $\downharpoonright f_1, f = f$.

(31) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence, then $\downharpoonleft f, f_{\mathrm{len}\,f} = f$.

(32) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $p \in \mathcal{L}(f_{\mathrm{Index}(p,f)}, f_{\mathrm{Index}(p,f)+1})$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose f is unfolded, s.n.c., and one-to-one and $\mathrm{len}\,f \geq 2$ and $f_1 \in \mathcal{L}(f,i)$. Then $i = 1$.

(34) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leq j$ and $j \leq \mathrm{width}\,\text{the Go-board of } f$ and $P = \mathcal{L}(\text{the Go-board of } f \circ (1, j), \text{the Go-board of } f \circ (\mathrm{len}\,\text{the Go-board of } f,\ j))$. Then P is a special polygonal arc joining the Go-board of $f \circ (1, j)$ and the Go-board of $f \circ (\mathrm{len}\,\text{the Go-board of } f,\ j)$.

(35) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leq j$ and $j \leq \mathrm{len}\,\text{the Go-board of } f$ and $P = \mathcal{L}(\text{the Go-board of } f \circ (j, 1), \text{the Go-board of } f \circ (j, \mathrm{width}\,\text{the Go-board of } f))$. Then P is a special polygonal arc joining the Go-board of $f \circ (j, 1)$ and the Go-board of $f \circ (j, \mathrm{width}\,\text{the Go-board of } f)$.

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Received September 10, 1997

Published January 2, 2004