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# Reconstructions of Special Sequences[1]

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**Summary.** We discuss here some methods for reconstructing special sequences which generate special polygonal arcs in $\mathcal{E}^2_{\mathrm{T}}$. For such reconstructions we introduce a "mid" function which cuts out the middle part of a sequence; the "$\downharpoonleft$" function, which cuts down the left part of a sequence at some point; the "$\downharpoonright$" function for cutting down the right part at some point; and the "$\downharpoonleft\downharpoonright$" function for cutting down both sides at two given points.

We also introduce some methods glueing two special sequences. By such cutting and glueing methods, the speciality of sequences (generatability of special polygonal arcs) is shown to be preserved.

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The articles [14], [17], [2], [3], [15], [10], [1], [11], [12], [18], [5], [4], [16], [6], [9], [8], [13], and [7] provide the notation and terminology for this paper.

## 1. Preliminaries

In this paper $i$, $i_1$, $i_2$, $n$ denote natural numbers.

Next we state a number of propositions:

(1) If $i -' i_1 \geq 1$ or $i - i_1 \geq 1$, then $i -' i_1 = i - i_1$.

(2) $n -' 0 = n$.

(3) $i_1 - i_2 \leq i_1 -' i_2$.

(4) If $i_1 \leq i_2$, then $n -' i_2 \leq n -' i_1$.

(5) If $i_1 \leq i_2$, then $i_1 -' n \leq i_2 -' n$.

(6) If $i -' i_1 \geq 1$ or $i - i_1 \geq 1$, then $(i -' i_1) + i_1 = i$.

(7) If $i_1 \leq i_2$, then $i_1 -' 1 \leq i_2$.

(8) $i -' 2 = i -' 1 -' 1$.

(9) If $i_1 + 1 \leq i_2$, then $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leq i_2$.

(10) Suppose $i_1 + 2 \leq i_2$ or $i_1 + 1 + 1 \leq i_2$. Then $i_1 + 1 < i_2$ and $(i_1 + 1) -' 1 < i_2$ and $(i_1 + 1) -' 2 < i_2$ and $i_1 + 1 \leq i_2$ and $(i_1 -' 1) + 1 < i_2$ and $((i_1 -' 1) + 1) -' 1 < i_2$ and $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leq i_2$.

[1]The work has been done while the second author was visiting Nagano in autumn 1996.

(11) If $i_1 \leq i_2$ or $i_1 \leq i_2 -' 1$, then $i_1 < i_2 + 1$ and $i_1 \leq i_2 + 1$ and $i_1 < i_2 + 1 + 1$ and $i_1 \leq i_2 + 1 + 1$ and $i_1 < i_2 + 2$ and $i_1 \leq i_2 + 2$.

(12) If $i_1 < i_2$ or $i_1 + 1 \leq i_2$, then $i_1 \leq i_2 -' 1$.

(13) If $i \geq i_1$, then $i \geq i_1 -' i_2$.

(14) If $1 \leq i$ and $1 \leq i_1 -' i$, then $i_1 -' i < i_1$.

We follow the rules: $n$, $i$, $i_1$, $j$ denote natural numbers and $D$ denotes a non empty set.
Next we state several propositions:

(15) For all finite sequences $p$, $q$ such that $\operatorname{len} p < i$ but $i \leq \operatorname{len} p + \operatorname{len} q$ or $i \leq \operatorname{len}(p \frown q)$ holds $(p \frown q)(i) = q(i - \operatorname{len} p)$.

(16) For every set $x$ and for every finite sequence $f$ holds $(f \frown \langle x \rangle)(\operatorname{len} f + 1) = x$ and $(\langle x \rangle \frown f)(1) = x$.

(17) Let $x$ be a set and $f$ be a finite sequence of elements of $D$. Suppose $1 \leq \operatorname{len} f$. Then $(f \frown \langle x \rangle)(1) = f(1)$ and $(f \frown \langle x \rangle)(1) = f_1$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f(\operatorname{len} f)$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f_{\operatorname{len} f}$.

(18) For every finite sequence $f$ such that $\operatorname{len} f = 1$ holds $\operatorname{Rev}(f) = f$.

(19) For every finite sequence $f$ of elements of $D$ and for every natural number $k$ holds $\operatorname{len}(f_{\downharpoonright k}) = \operatorname{len} f -' k$.

(20) Let $D$ be a set, $f$ be a finite sequence of elements of $D$, and $k$ be a natural number. If $k \leq n$, then $(f \upharpoonright n)(k) = f(k)$.

(21) For every finite sequence $f$ of elements of $D$ and for all natural numbers $l_1$, $l_2$ holds $f_{\downharpoonright l_1} \upharpoonright (l_2 -' l_1) = (f \upharpoonright l_2)_{\downharpoonright l_1}$.

## 2. Middle Function for Finite Sequences

Let us consider $D$, let $f$ be a finite sequence of elements of $D$, and let $k_1$, $k_2$ be natural numbers. The functor $\operatorname{mid}(f, k_1, k_2)$ yielding a finite sequence of elements of $D$ is defined as follows:

(Def. 1) $$\operatorname{mid}(f, k_1, k_2) = \begin{cases} f_{\downharpoonright k_1 -' 1} \upharpoonright ((k_2 -' k_1) + 1), & \text{if } k_1 \leq k_2, \\ \operatorname{Rev}(f_{\downharpoonright k_2 -' 1} \upharpoonright ((k_1 -' k_2) + 1)), & \text{otherwise.} \end{cases}$$

The following propositions are true:

(22) Let $f$ be a finite sequence of elements of $D$ and $k_1$, $k_2$ be natural numbers. If $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$, then $\operatorname{Rev}(\operatorname{mid}(f, k_1, k_2)) = \operatorname{mid}(\operatorname{Rev}(f), (\operatorname{len} f -' k_2) + 1, (\operatorname{len} f -' k_1) + 1)$.

(23) Let $n$, $m$ be natural numbers and $f$ be a finite sequence of elements of $D$. If $1 \leq m$ and $m + n \leq \operatorname{len} f$, then $f_{\downharpoonright n}(m) = f(m + n)$.

(24) Let $i$ be a natural number and $f$ be a finite sequence of elements of $D$. If $1 \leq i$ and $i \leq \operatorname{len} f$, then $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(25) For every finite sequence $f$ of elements of $D$ and for every natural number $k$ such that $1 \leq k$ holds $\operatorname{mid}(f, 1, k) = f \upharpoonright k$.

(26) For every finite sequence $f$ of elements of $D$ and for every natural number $k$ such that $k \leq \operatorname{len} f$ holds $\operatorname{mid}(f, k, \operatorname{len} f) = f_{\downharpoonright k -' 1}$.

(27) Let $f$ be a finite sequence of elements of $D$ and $k_1$, $k_2$ be natural numbers. Suppose $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$. Then

(i) $(\operatorname{mid}(f,k_1,k_2))(1) = f(k_1)$,

(ii) if $k_1 \leq k_2$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) = (k_2 -' k_1)+1$ and for every natural number $i$ such that $1 \leq i$ and $i \leq \operatorname{len}\operatorname{mid}(f,k_1,k_2)$ holds $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1)-' 1)$, and

(iii) if $k_1 > k_2$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) = (k_1 -' k_2)+1$ and for every natural number $i$ such that $1 \leq i$ and $i \leq \operatorname{len}\operatorname{mid}(f,k_1,k_2)$ holds $(\operatorname{mid}(f,k_1,k_2))(i) = f((k_1 -' i)+1)$.

(28) For every finite sequence $f$ of elements of $D$ and for all natural numbers $k_1$, $k_2$ holds $\operatorname{rng}\operatorname{mid}(f,k_1,k_2) \subseteq \operatorname{rng} f$.

(29) For every finite sequence $f$ of elements of $D$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{mid}(f,1,\operatorname{len} f) = f$.

(30) For every finite sequence $f$ of elements of $D$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{mid}(f,\operatorname{len} f,1) = \operatorname{Rev}(f)$.

(31) Let $f$ be a finite sequence of elements of $D$ and $k_1$, $k_2$, $i$ be natural numbers. Suppose $1 \leq k_1$ and $k_1 \leq k_2$ and $k_2 \leq \operatorname{len} f$ and $1 \leq i$ and $i \leq (k_2 -' k_1)+1$ or $i \leq (k_2 - k_1)+1$ or $i \leq (k_2 + 1) - k_1$. Then $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1)-' 1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i-' 1)+k_1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1)-1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i-1)+k_1)$.

(32) Let $f$ be a finite sequence of elements of $D$ and $k$, $i$ be natural numbers. If $1 \leq i$ and $i \leq k$ and $k \leq \operatorname{len} f$, then $(\operatorname{mid}(f,1,k))(i) = f(i)$.

(33) Let $f$ be a finite sequence of elements of $D$ and $k_1$, $k_2$ be natural numbers. If $1 \leq k_1$ and $k_1 \leq k_2$ and $k_2 \leq \operatorname{len} f$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) \leq \operatorname{len} f$.

(34) For every finite sequence $f$ of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $2 \leq \operatorname{len} f$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $f_1 \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \in \widetilde{\mathcal{L}}(f)$.

(35) For all points $p_1$, $p_2$, $q_1$, $q_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $q_1 \in \mathcal{L}(p_1,p_2)$ but $q_2 \in \mathcal{L}(p_1,p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(36) For all points $p_1$, $p_2$, $q_1$, $q_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $\mathcal{L}(q_1,q_2) \subseteq \mathcal{L}(p_1,p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(37) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $n$ be a natural number. If $2 \leq n$ and $f$ is a special sequence, then $f {\restriction} n$ is a special sequence.

(38) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $n$ be a natural number. Suppose $n \leq \operatorname{len} f$ and $2 \leq \operatorname{len} f -' n$ and $f$ is a special sequence. Then $f_{\downharpoonleft n}$ is a special sequence.

(39) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $k_1$, $k_2$ be natural numbers. Suppose $f$ is a special sequence and $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$ and $k_1 \neq k_2$. Then $\operatorname{mid}(f,k_1,k_2)$ is a special sequence.

## 3. A Concept of Index for Finite Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that $p \in \widetilde{\mathcal{L}}(f)$. The functor $\operatorname{Index}(p,f)$ yielding a natural number is defined by:

(Def. 2) There exists a non empty subset $S$ of $\mathbb{N}$ such that $\operatorname{Index}(p,f) = \min S$ and $S = \{i : p \in \mathcal{L}(f,i)\}$.

One can prove the following propositions:

(40) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$, and $i$ be a natural number. If $p \in \mathcal{L}(f,i)$, then $\operatorname{Index}(p,f) \leq i$.

(41) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $1 \leq \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) < \mathrm{len}\, f$.

(42) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$.

(43) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \mathcal{L}(f, 1)$ holds $\mathrm{Index}(p, f) = 1$.

(44) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $\mathrm{len}\, f \geq 2$ holds $\mathrm{Index}(f_1, f) = 1$.

(45) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, $p$ be a point of $\mathcal{E}^2_{\rm T}$, and given $i_1$. If $f$ is a special sequence and $1 < i_1$ and $i_1 \leq \mathrm{len}\, f$ and $p = f(i_1)$, then $\mathrm{Index}(p, f) + 1 = i_1$.

(46) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, $p$ be a point of $\mathcal{E}^2_{\rm T}$, and given $i_1$. If $f$ is a special sequence and $p \in \mathcal{L}(f, i_1)$, then $i_1 = \mathrm{Index}(p, f)$ or $i_1 = \mathrm{Index}(p, f) + 1$.

(47) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, $p$ be a point of $\mathcal{E}^2_{\rm T}$, and given $i_1$. If $f$ is a special sequence and $i_1 + 1 \leq \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i_1)$ and $p \neq f(i_1)$, then $i_1 = \mathrm{Index}(p, f)$.

Let $g$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\rm T}$. We say that $g$ is a special sequence joining $p_1$, $p_2$ if and only if:

(Def. 3) $g$ is a special sequence and $g(1) = p_1$ and $g(\mathrm{len}\, g) = p_2$.

One can prove the following propositions:

(48) Let $g$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p_1$, $p_2$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $g$ is a special sequence joining $p_1$, $p_2$. Then $\mathrm{Rev}(g)$ is a special sequence joining $p_2$, $p_1$.

(49) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, $p$ be a point of $\mathcal{E}^2_{\rm T}$, and given $j$. If $p \in \widetilde{\mathcal{L}}(f)$ and $g = \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f)$ and $1 \leq j$ and $j + 1 \leq \mathrm{len}\, g$, then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, (\mathrm{Index}(p, f) + j) -' 1)$.

(50) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{Index}(p, f) + 1)$ and $g = \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f)$. Then $g$ is a special sequence joining $p$, $f_{\mathrm{len}\, f}$.

(51) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, $p$ be a point of $\mathcal{E}^2_{\rm T}$, and given $j$. If $p \in \widetilde{\mathcal{L}}(f)$ and $1 \leq j$ and $j + 1 \leq \mathrm{len}\, g$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$, then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, j)$.

(52) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$. Then $g$ is a special sequence joining $f_1$, $p$.

## 4. Left and Right Cutting Functions for Finite Sequences in $\mathcal{E}^2_{\rm T}$

Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let $p$ be a point of $\mathcal{E}^2_{\rm T}$. The functor $\downharpoonleft p, f$ yielding a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 4) $$\downharpoonleft p, f = \begin{cases} \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f), \text{ if } p \neq f(\mathrm{Index}(p, f) + 1), \\ \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f), \text{ otherwise.} \end{cases}$$

The functor $\downharpoonright f, p$ yields a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 5) $$\downharpoonright f, p = \begin{cases} (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle, \text{ if } p \neq f(1), \\ \langle p \rangle, \text{ otherwise.} \end{cases}$$

One can prove the following propositions:

(53) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p = f(\mathrm{Index}(p, f) + 1)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) + 1 = \mathrm{len}\, f$.

(54) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. If $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) = \mathrm{len}\, f$.

(55) Let given $D$, $f$ be a finite sequence of elements of $D$, $k$ be a natural number, and $p$ be an element of $D$. Then $(\langle p \rangle \frown f) \restriction (k + 1) = \langle p \rangle \frown (f \restriction k)$.

(56) Let given $D$, $f$ be a non empty finite sequence of elements of $D$, and $k_1$, $k_2$ be natural numbers. If $k_1 < k_2$ and $k_1 \in \mathrm{dom}\, f$, then $\mathrm{mid}(f, k_1, k_2) = \langle f(k_1) \rangle \frown \mathrm{mid}(f, k_1 + 1, k_2)$.

Let $f$ be a non empty finite sequence. One can check that $\mathrm{Rev}(f)$ is non empty.

One can prove the following propositions:

(57) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. If $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright p, \mathrm{Rev}(f) = \mathrm{Rev}(\downharpoonleft f, p)$.

(58) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $p \in \widetilde{\mathcal{L}}(f)$. Then

(i) $(\downharpoonright p, f)(1) = p$, and

(ii) for every $i$ such that $1 < i$ and $i \leq \mathrm{len} \downharpoonright p, f$ holds if $p = f(\mathrm{Index}(p, f) + 1)$, then $(\downharpoonright p, f)(i) = f(\mathrm{Index}(p, f) + i)$ and if $p \neq f(\mathrm{Index}(p, f) + 1)$, then $(\downharpoonright p, f)(i) = f((\mathrm{Index}(p, f) + i) - 1)$.

(59) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then $(\downharpoonleft f, p)(\mathrm{len} \downharpoonleft f, p) = p$ and for every $i$ such that $1 \leq i$ and $i \leq \mathrm{Index}(p, f)$ holds $(\downharpoonleft f, p)(i) = f(i)$.

(60) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$ such that $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then

(i) if $p \neq f(1)$, then $\mathrm{len} \downharpoonleft f, p = \mathrm{Index}(p, f) + 1$, and

(ii) if $p = f(1)$, then $\mathrm{len} \downharpoonleft f, p = \mathrm{Index}(p, f)$.

(61) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$ such that $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then

(i) if $p = f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{len} \downharpoonright p, f = \mathrm{len}\, f - \mathrm{Index}(p, f)$, and

(ii) if $p \neq f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{len} \downharpoonright p, f = (\mathrm{len}\, f - \mathrm{Index}(p, f)) + 1$.

Let $p_1$, $p_2$, $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. The predicate $q_1 \leq_{p_1, p_2} q_2$ is defined by the conditions (Def. 6).

(Def. 6)(i) $q_1 \in \mathcal{L}(p_1, p_2)$,

(ii) $q_2 \in \mathcal{L}(p_1, p_2)$, and

(iii) for all real numbers $r_1$, $r_2$ such that $0 \leq r_1$ and $r_1 \leq 1$ and $q_1 = (1 - r_1) \cdot p_1 + r_1 \cdot p_2$ and $0 \leq r_2$ and $r_2 \leq 1$ and $q_2 = (1 - r_2) \cdot p_1 + r_2 \cdot p_2$ holds $r_1 \leq r_2$.

Let $p_1$, $p_2$, $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. The predicate $q_1 <_{p_1, p_2} q_2$ is defined as follows:

(Def. 7) $q_1 \leq_{p_1, p_2} q_2$ and $q_1 \neq q_2$.

Next we state several propositions:

(62) For all points $p_1$, $p_2$, $q_1$, $q_2$ of $\mathcal{E}^2_{\rm T}$ such that $q_1 \leq_{p_1, p_2} q_2$ and $q_2 \leq_{p_1, p_2} q_1$ holds $q_1 = q_2$.

(63) For all points $p_1$, $p_2$, $q_1$, $q_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $q_1 \leq_{p_1,p_2} q_2$ or $q_2 <_{p_1,p_2} q_1$ but $q_1 \leq_{p_1,p_2} q_2$ but $q_2 <_{p_1,p_2} q_1$.

(64) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$, $q$, $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$, then $q \in \widetilde{\mathcal{L}}(\downharpoonleft p, f)$.

(65) For all points $p$, $q$, $p_1$, $p_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \leq_{p_1,p_2} q$ holds $q \in \mathcal{L}(p, p_2)$ and $p \in \mathcal{L}(p_1, q)$.

(66) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$, $q$, $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $p \leq_{f_{\mathrm{Index}(p,f)}, f_{\mathrm{Index}(p,f)+1}} q$. Then $q \in \widetilde{\mathcal{L}}(\downharpoonleft p, f)$.

## 5. Cutting Both Sides of a Finite Sequence and a Discussion of Speciality of Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let $p$, $q$ be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\downharpoonleft\downharpoonright p, f, q$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 8) $$\downharpoonleft\downharpoonright p, f, q = \begin{cases} \downharpoonright\downharpoonleft p, f, q, \text{ if } p \in \widetilde{\mathcal{L}}(f) \text{ and } q \in \widetilde{\mathcal{L}}(f) \text{ and } \mathrm{Index}(p, f) < \mathrm{Index}(q, f) \text{ or } \mathrm{Index}(p, f) = \mathrm{Index}(q, f) \text{ and } p \leq \\ \mathrm{Rev}(\downharpoonright\downharpoonleft q, f, p), \text{ otherwise.} \end{cases}$$

One can prove the following propositions:

(67) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonright f, p$ is a special sequence joining $f_1$, $p$.

(68) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonleft p, f$ is a special sequence joining $p$, $f_{\mathrm{len}\, f}$.

(69) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonleft p, f$ is a special sequence.

(70) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonright f, p$ is a special sequence.

(71) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$, $q$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\downharpoonleft\downharpoonright p, f, q$ is a special sequence joining $p$, $q$.

(72) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$, $q$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\downharpoonleft\downharpoonright p, f, q$ is a special sequence.

(73) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence.

(74) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence joining $f_1$, $g_{\mathrm{len}\, g}$.

(75) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number $n$ holds $\widetilde{\mathcal{L}}(f_{\restriction n}) \subseteq \widetilde{\mathcal{L}}(f)$.

(76) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $f$ is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonright f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(77) Let $f$ be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $f$ is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonleft p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(78) Let $f$, $g$ be non empty finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\downharpoonleft p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence joining $p$, $g_{\operatorname{len} g}$.

(79) Let $f$, $g$ be non empty finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\downharpoonleft p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence.

(80) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence.

(81) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence joining $f_1$, $g_{\operatorname{len} g}$.

(82) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonright g, p$ is a special sequence joining $f_1$, $p$.

(83) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and $f$ is a special sequence and $g$ is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonright g, p$ is a special sequence.

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