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On the Minimal Distance Between Sets in Euclidean Space[1]

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Summary. The concept of the minimal distance between two sets in a Euclidean space is introduced and some useful lemmas are proved.

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The articles [27], [29], [1], [28], [15], [26], [5], [30], [7], [6], [2], [16], [19], [25], [10], [24], [17], [8], [4], [11], [12], [13], [3], [21], [23], [14], [22], [9], [18], and [20] provide the notation and terminology for this paper.

1. Preliminaries

In this paper X denotes a set and Y denotes a non empty set.

Next we state several propositions:

(1) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists a set x such that $x \in X$ and $y = f(x)$.

(2) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists an element x of X such that $y = f(x)$.

(3) For every function f from X into Y and for every subset A of X such that f is onto holds $(f^\circ A)^{\rm c} \subseteq f^\circ A^{\rm c}$.

(4) For every function f from X into Y and for every subset A of X such that f is one-to-one holds $f^\circ A^{\rm c} \subseteq (f^\circ A)^{\rm c}$.

(5) For every function f from X into Y and for every subset A of X such that f is bijective holds $(f^\circ A)^{\rm c} = f^\circ A^{\rm c}$.

2. Topological and Metrizable Spaces

We now state two propositions:

(6) For every topological space T and for every subset A of T holds A is a component of $\emptyset_T$ iff A is empty.

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(7) Let T be a non empty topological space and A, B, C be subsets of T. If $A \subseteq B$ and A is a component of C and B is a component of C, then $A = B$.

In the sequel n is a natural number.
The following proposition is true

(8) If $n \geq 1$, then for every subset P of $\mathcal{E}^n$ such that P is bounded holds $P^{\rm c}$ is not bounded.

In the sequel r is a real number and M is a non empty metric space.
Next we state several propositions:

(9) For every non empty subset C of $M_{\rm top}$ and for every point p of $M_{\rm top}$ holds $({\rm dist}_{\rm min}(C))(p) \geq 0$.

(10) Let C be a non empty subset of $M_{\rm top}$ and p be a point of M. If for every point q of M such that $q \in C$ holds $\rho(p,q) \geq r$, then $({\rm dist}_{\rm min}(C))(p) \geq r$.

(11) For all non empty subsets A, B of $M_{\rm top}$ holds ${\rm dist}^{\rm min}_{\rm min}(A,B) \geq 0$.

(12) For all compact subsets A, B of $M_{\rm top}$ such that A meets B holds ${\rm dist}^{\rm min}_{\rm min}(A,B) = 0$.

(13) Let A, B be non empty subsets of $M_{\rm top}$. Suppose that for all points p, q of M such that $p \in A$ and $q \in B$ holds $\rho(p,q) \geq r$. Then ${\rm dist}^{\rm min}_{\rm min}(A,B) \geq r$.

3. Euclid Topological Spaces

We now state several propositions:

(14) Let P, Q be subsets of $\mathcal{E}^n_{\rm T}$. Suppose P is a component of $Q^{\rm c}$. Then P is inside component of Q or P is outside component of Q.

(15) If $n \geq 1$, then ${\rm BDD}\, \emptyset_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(16) ${\rm BDD}\, \Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(17) If $n \geq 1$, then ${\rm UBD}\, \emptyset_{\mathcal{E}^n_{\rm T}} = \Omega_{\mathcal{E}^n_{\rm T}}$.

(18) ${\rm UBD}\, \Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(19) For every connected subset P of $\mathcal{E}^n_{\rm T}$ and for every subset Q of $\mathcal{E}^n_{\rm T}$ such that P misses Q holds $P \subseteq {\rm UBD}\, Q$ or $P \subseteq {\rm BDD}\, Q$.

4. Euclid Plane

For simplicity, we follow the rules: C, D denote simple closed curves, n denotes a natural number, p, q, q_1, q_2 denote points of $\mathcal{E}^2_{\rm T}$, r, s_1, s_2, t_1, t_2 denote real numbers, and x, y denote points of $\mathcal{E}^2$.

The following propositions are true:

(20) $\rho([0,0], r \cdot q) = |r| \cdot \rho([0,0], q)$.

(21) $\rho(q_1+q, q_2+q) = \rho(q_1,q_2)$.

(22) If $p \neq q$, then $\rho(p,q) > 0$.

(23) $\rho(q_1-q, q_2-q) = \rho(q_1,q_2)$.

(24) $\rho(p,q) = \rho(-p,-q)$.

(25) $\rho(q-q_1, q-q_2) = \rho(q_1,q_2)$.

(26) $\rho(r \cdot p, r \cdot q) = |r| \cdot \rho(p,q)$.

(27) If $r \leq 1$, then $\rho(p, r \cdot p + (1-r) \cdot q) = (1-r) \cdot \rho(p,q)$.

(28) If $0 \leq r$, then $\rho(q, r \cdot p + (1-r) \cdot q) = r \cdot \rho(p,q)$.

(29) If $p \in \mathcal{L}(q_1, q_2)$, then $\rho(q_1, p) + \rho(p, q_2) = \rho(q_1, q_2)$.

(30) If $q_1 \in \mathcal{L}(q_2, p)$ and $q_1 \neq q_2$, then $\rho(q_1, p) < \rho(q_2, p)$.

(31) If $y = [0,0]$, then $\mathrm{Ball}(y,r) = \{q : |q| < r\}$.

5. Affine Maps

Next we state several propositions:

(32) $(\mathrm{AffineMap}(r, s_1, r, s_2))(p) = r \cdot p + [s_1, s_2]$.

(33) $(\mathrm{AffineMap}(r, q_{\mathbf{1}}, r, q_{\mathbf{2}}))(p) = r \cdot p + q$.

(34) If $s_1 > 0$ and $s_2 > 0$, then $\mathrm{AffineMap}(s_1, t_1, s_2, t_2) \cdot \mathrm{AffineMap}(\frac{1}{s_1}, -\frac{t_1}{s_1}, \frac{1}{s_2}, -\frac{t_2}{s_2}) = \mathrm{id}_{\mathcal{R}^2}$.

(35) If $y = [0,0]$ and $x = q$ and $r > 0$, then $(\mathrm{AffineMap}(r, q_{\mathbf{1}}, r, q_{\mathbf{2}}))^{\circ}\,\mathrm{Ball}(y,1) = \mathrm{Ball}(x,r)$.

(36) For all real numbers A, B, C, D such that $A > 0$ and $C > 0$ holds $\mathrm{AffineMap}(A,B,C,D)$ is onto.

(37) $\mathrm{Ball}(x,r)^{\mathrm{c}}$ is a connected subset of $\mathcal{E}^2_{\mathrm{T}}$.

6. Minimal Distance Between Subsets

Let us consider n and let A, B be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{dist}_{\min}(A,B)$ yields a real number and is defined as follows:

(Def. 1) There exist subsets A', B' of $(\mathcal{E}^n)_{\mathrm{top}}$ such that $A = A'$ and $B = B'$ and $\mathrm{dist}_{\min}(A,B) = \mathrm{dist}^{\min}_{\min}(A', B')$.

Let M be a non empty metric space and let P, Q be non empty compact subsets of M_{top}. Let us note that the functor $\mathrm{dist}^{\min}_{\min}(P,Q)$ is commutative. Let us note that the functor $\mathrm{dist}^{\max}_{\max}(P,Q)$ is commutative.

Let us consider n and let A, B be non empty compact subsets of $\mathcal{E}^n_{\mathrm{T}}$. Let us note that the functor $\mathrm{dist}_{\min}(A,B)$ is commutative.

The following propositions are true:

(38) For all non empty subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{dist}_{\min}(A,B) \geq 0$.

(39) For all compact subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that A meets B holds $\mathrm{dist}_{\min}(A,B) = 0$.

(40) Let A, B be non empty subsets of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ and $q \in B$ holds $\rho(p,q) \geq r$. Then $\mathrm{dist}_{\min}(A,B) \geq r$.

(41) For every subset D of $\mathcal{E}^n_{\mathrm{T}}$ and for all non empty subsets A, C of $\mathcal{E}^n_{\mathrm{T}}$ such that $C \subseteq D$ holds $\mathrm{dist}_{\min}(A,D) \leq \mathrm{dist}_{\min}(A,C)$.

(42) For all non empty compact subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ there exist points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ and $q \in B$ and $\mathrm{dist}_{\min}(A,B) = \rho(p,q)$.

(43) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{dist}_{\min}(\{p\},\{q\}) = \rho(p,q)$.

Let us consider n, let p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\rho(p,B)$ yields a real number and is defined by:

(Def. 2) $\rho(p,B) = \mathrm{dist}_{\min}(\{p\},B)$.

One can prove the following propositions:

(44) For every non empty subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every point p of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(p,A) \geq 0$.

(45) For every compact subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every point p of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ holds $\rho(p,A) = 0$.

(46) Let A be a non empty compact subset of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a point q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ and $\rho(p,A) = \rho(p,q)$.

(47) Let C be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$ and D be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $C \subseteq D$, then for every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(q,D) \leq \rho(q,C)$.

(48) Let A be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. If for every point q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $\rho(p,q) \geq r$, then $\rho(p,A) \geq r$.

(49) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(p,\{q\}) = \rho(p,q)$.

(50) For every non empty subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $\rho(p,A) \leq \rho(p,q)$.

(51) Let A be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and B be an open subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A \subseteq B$, then for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \notin B$ holds $\rho(p,B) < \rho(p,A)$.

7. BDD AND UBD

Next we state two propositions:

(52) $\mathrm{UBD}\,C$ meets $\mathrm{UBD}\,D$.

(53) If $q \in \mathrm{UBD}\,C$ and $p \in \mathrm{BDD}\,C$, then $\rho(q,C) < \rho(q,p)$.

Let us consider C. Note that $\mathrm{BDD}\,C$ is non empty.
The following three propositions are true:

(54) If $p \notin \mathrm{BDD}\,C$, then $\rho(p,C) \leq \rho(p,\mathrm{BDD}\,C)$.

(55) $C \not\subseteq \mathrm{BDD}\,D$ or $D \not\subseteq \mathrm{BDD}\,C$.

(56) If $C \subseteq \mathrm{BDD}\,D$, then $D \subseteq \mathrm{UBD}\,C$.

8. MAIN DEFINITIONS

We now state the proposition

(57) $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \subseteq \mathrm{UBD}\,C$.

Let us consider C. The functor LowerMiddlePoint$C$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 3) $\mathrm{LowerMiddlePoint}\,C = \mathrm{FPoint}(\mathrm{LowerArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$.

The functor UpperMiddlePoint$C$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $\mathrm{UpperMiddlePoint}\,C = \mathrm{FPoint}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$.

Next we state several propositions:

(58) $\mathrm{LowerArc}(C)$ meets $\mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$.

(59) $\mathrm{UpperArc}(C)$ meets $\mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$.

(60) $(\mathrm{LowerMiddlePoint}\,C)_{\mathbf{1}} = \frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}$.

(61) $(\mathrm{UpperMiddlePoint}\,C)_{\mathbf{1}} = \frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}$.

(62) $\mathrm{LowerMiddlePoint}\,C \in \mathrm{LowerArc}(C)$.

(63) $\mathrm{UpperMiddlePoint}\,C \in \mathrm{UpperArc}(C)$.

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