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More on External Approximation of a Continuum[1]

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Summary. The main goal was to prove two facts:

- the gauge is the Go-Board of a corresponding cage,
- the left components of the complement of the curve determined by a cage are monotonic wrt the index of the approximation.

Some auxiliary facts are proved, too. At the end the new notion needed for the internal approximation are defined and some useful lemmas are proved.

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The articles [37], [10], [46], [39], [12], [32], [3], [40], [22], [2], [42], [34], [47], [49], [48], [7], [9], [8], [1], [4], [19], [5], [11], [44], [13], [23], [24], [43], [45], [28], [36], [50], [18], [35], [6], [20], [30], [41], [21], [26], [27], [31], [33], [29], [16], [38], [14], [15], [17], and [25] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: m, k, j, j_1, i, i_1, i_2, n are natural numbers, r, s are real numbers, C is a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$, G is a Go-board, and p is a point of $\mathcal{E}^2_{\mathrm{T}}$.

Let us observe that there exists a set which has a non-empty element.

Let D be a non empty set with a non-empty element. Observe that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let D be a non empty set with non empty elements. Note that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let F be a non-empty function yielding function. One can verify that $\mathrm{rng}_\kappa F(\kappa)$ is non-empty.

One can verify that every finite sequence of elements of $\mathbb{R}$ which is increasing is also one-to-one.

One can prove the following propositions:

(4)[1] For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is convex.

(5) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is connected.

(6) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $p \in \overline{\mathcal{L}(p,q) \setminus \{p,q\}}$.

(7) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $\overline{\mathcal{L}(p,q) \setminus \{p,q\}} = \mathcal{L}(p,q)$.

(8) For every subset S of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ and $\mathcal{L}(p,q) \setminus \{p,q\} \subseteq S$ holds $\mathcal{L}(p,q) \subseteq \overline{S}$.

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[1] The propositions (1)–(3) have been removed.

2. Transforming Finite Sets to Finite Sequences

The binary relation RealOrd on $\mathbb{R}$ is defined by:

(Def. 1) $\text{RealOrd} = \{\langle r, s\rangle : r \leq s\}$.

We now state two propositions:

(9) If $\langle r, s\rangle \in \text{RealOrd}$, then $r \leq s$.

(10) $\text{field}\,\text{RealOrd} = \mathbb{R}$.

One can check that RealOrd is total, reflexive, antisymmetric, transitive, and linear-order.
The following three propositions are true:

(11) RealOrd linearly orders $\mathbb{R}$.

(12) For every finite subset A of $\mathbb{R}$ holds $\text{SgmX}(\text{RealOrd}, A)$ is increasing.

(13) For every finite sequence f of elements of $\mathbb{R}$ and for every finite subset A of $\mathbb{R}$ such that $A = \text{rng}\, f$ holds $\text{SgmX}(\text{RealOrd}, A) = \text{Inc}(f)$.

Let A be a finite subset of $\mathbb{R}$. One can check that $\text{SgmX}(\text{RealOrd}, A)$ is increasing.
We now state the proposition

(15)[2] For every non empty set X and for every finite subset A of X and for every linear-order order R in X holds $\text{len}\,\text{SgmX}(R, A) = \text{card}\, A$.

3. On the Construction of Go-Boards

The following two propositions are true:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{X}\text{-coordinate}(f) = \text{proj1} \cdot f$.

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{Y}\text{-coordinate}(f) = \text{proj2} \cdot f$.

Let D be a non empty set and let M be a finite sequence of elements of D^*. Then $\text{Values}\, M$ is a subset of D.

Let D be a non empty set with non empty elements and let M be a non empty non-empty finite sequence of elements of D^*. One can verify that $\text{Values}\, M$ is non empty.

The following propositions are true:

(18) For every non empty set D and for every matrix M over D and for every i such that $i \in \text{Seg}\,\text{width}\, M$ holds $\text{rng}(M_{\square,i}) \subseteq \text{Values}\, M$.

(19) For every non empty set D and for every matrix M over D and for every i such that $i \in \text{dom}\, M$ holds $\text{rng}\,\text{Line}(M, i) \subseteq \text{Values}\, M$.

(20) For every column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{len}\, G \leq \text{card}(\text{proj1}^\circ\, \text{Values}\, G)$.

(21) For every line $\mathbf{X}$-constant matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{card}(\text{proj1}^\circ\, \text{Values}\, G) \leq \text{len}\, G$.

(22) For every line $\mathbf{X}$-constant column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{len}\, G = \text{card}(\text{proj1}^\circ\, \text{Values}\, G)$.

(23) For every line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{width}\, G \leq \text{card}(\text{proj2}^\circ\, \text{Values}\, G)$.

(24) For every column $\mathbf{Y}$-constant non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{card}(\text{proj2}^\circ\, \text{Values}\, G) \leq \text{width}\, G$.

(25) For every column $\mathbf{Y}$-constant line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{width}\, G = \text{card}(\text{proj2}^\circ\, \text{Values}\, G)$.

[2] The proposition (14) has been removed.

4. More about Go-Boards

Next we state several propositions:

(26) Let G be a Go-board and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G. Let k be a natural number. If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\mathcal{L}(f,k) \subseteq \mathrm{left\_cell}(f,k,G)$.

(27) For every standard special circular sequence f such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ holds $\mathrm{left\_cell}(f,k,\text{the Go-board of } f) = \mathrm{leftcell}(f,k)$.

(28) Let G be a Go-board and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G. Let k be a natural number. If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\mathcal{L}(f,k) \subseteq \mathrm{right\_cell}(f,k,G)$.

(29) For every standard special circular sequence f such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ holds $\mathrm{right\_cell}(f,k,\text{the Go-board of } f) = \mathrm{rightcell}(f,k)$.

(30) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a non constant standard special circular sequence. If P is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$, then $P = \mathrm{RightComp}(f)$ or $P = \mathrm{LeftComp}(f)$.

(31) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\operatorname{Int}\mathrm{right\_cell}(f,k,G) \subseteq \mathrm{RightComp}(f)$ and $\operatorname{Int}\mathrm{left\_cell}(f,k,G) \subseteq \mathrm{LeftComp}(f)$.

(32) Let i_1, j_1, i_2, j_2 be natural numbers and G be a Go-board. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G and $G \circ (i_1, j_1) = G \circ (i_2, j_2)$. Then $i_1 = i_2$ and $j_1 = j_2$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and M be a Go-board. Suppose f is a sequence which elements belong to M. Then $\mathrm{mid}(f, i_1, i_2)$ is a sequence which elements belong to M.

Let us note that every Go-board is non empty and non-empty.

We now state four propositions:

(34) For every Go-board G such that $1 \leq i$ and $i \leq \operatorname{len} G$ holds $(\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj1}^\circ \mathrm{Values}\, G))(i) = (G \circ (i,1))_{\mathbf{1}}$.

(35) For every Go-board G such that $1 \leq j$ and $j \leq \operatorname{width} G$ holds $(\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj2}^\circ \mathrm{Values}\, G))(j) = (G \circ (1,j))_{\mathbf{2}}$.

(36) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i\rangle \in$ the indices of G and $G \circ (1,i) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle \operatorname{len} G, i\rangle \in$ the indices of G and $G \circ (\operatorname{len} G, i) \in \operatorname{rng} f$.

Then $\mathrm{proj1}^\circ \operatorname{rng} f = \mathrm{proj1}^\circ \mathrm{Values}\, G$.

(37) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle i, 1\rangle \in$ the indices of G and $G \circ (i,1) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle i, \operatorname{width} G\rangle \in$ the indices of G and $G \circ (i, \operatorname{width} G) \in \operatorname{rng} f$.

Then $\mathrm{proj2}^\circ \operatorname{rng} f = \mathrm{proj2}^\circ \mathrm{Values}\, G$.

Let G be a Go-board. One can verify that $\mathrm{Values}\, G$ is non empty.

Next we state three propositions:

(38) For every Go-board G holds $G =$ the Go-board of $\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj1}^\circ \mathrm{Values}\, G)$, $\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj2}^\circ \mathrm{Values}\, G)$.

(39) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. If $\mathrm{proj1}^\circ \mathrm{rng} f = \mathrm{proj1}^\circ \mathrm{Values}\, G$ and $\mathrm{proj2}^\circ \mathrm{rng} f = \mathrm{proj2}^\circ \mathrm{Values}\, G$, then $G =$ the Go-board of f.

(40) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i \rangle \in$ the indices of G and $G \circ (1,i) \in \mathrm{rng} f$,

(iii) there exists i such that $\langle i, 1 \rangle \in$ the indices of G and $G \circ (i,1) \in \mathrm{rng} f$,

(iv) there exists i such that $\langle \mathrm{len}\, G, i \rangle \in$ the indices of G and $G \circ (\mathrm{len}\, G, i) \in \mathrm{rng} f$, and

(v) there exists i such that $\langle i, \mathrm{width}\, G \rangle \in$ the indices of G and $G \circ (i, \mathrm{width}\, G) \in \mathrm{rng} f$.

Then $G =$ the Go-board of f.

5. More about Gauges

We now state several propositions:

(41) If $m \leq n$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, \mathrm{Gauge}(C,n)$, then $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ is a natural number.

(42) If $m \leq n$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, \mathrm{Gauge}(C,n)$, then $1 \leq \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor + 1 \leq \mathrm{len}\, \mathrm{Gauge}(C,m)$.

(43) Suppose $m \leq n$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, \mathrm{Gauge}(C,n)$ and $1 \leq j$ and $j+1 \leq \mathrm{width}\, \mathrm{Gauge}(C,n)$. Then there exist i_1, j_1 such that $i_1 = \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $j_1 = \lfloor \frac{j-2}{2^{n-'m}} + 2 \rfloor$ and $\mathrm{cell}(\mathrm{Gauge}(C,n), i, j) \subseteq \mathrm{cell}(\mathrm{Gauge}(C,m), i_1, j_1)$.

(44) Suppose $m \leq n$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, \mathrm{Gauge}(C,n)$ and $1 \leq j$ and $j+1 \leq \mathrm{width}\, \mathrm{Gauge}(C,n)$. Then there exist i_1, j_1 such that $1 \leq i_1$ and $i_1 + 1 \leq \mathrm{len}\, \mathrm{Gauge}(C,m)$ and $1 \leq j_1$ and $j_1 + 1 \leq \mathrm{width}\, \mathrm{Gauge}(C,m)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n), i, j) \subseteq \mathrm{cell}(\mathrm{Gauge}(C,m), i_1, j_1)$.

(47)[3] For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is Bounded holds $\mathrm{UBD}\, P$ is not Bounded.

(48) Let f be a non constant standard special circular sequence. If $f \circlearrowleft p$ is clockwise oriented, then f is clockwise oriented.

(49) For every non constant standard special circular sequence f such that $\mathrm{LeftComp}(f) = \mathrm{UBD}\, \widetilde{\mathcal{L}}(f)$ holds f is clockwise oriented.

6. More about Cages

We now state several propositions:

(50) For every non constant standard special circular sequence f holds $\overline{\mathrm{LeftComp}(f)^{\mathrm{c}}} = \mathrm{RightComp}(f)$.

(51) For every non constant standard special circular sequence f holds $\overline{\mathrm{RightComp}(f)^{\mathrm{c}}} = \mathrm{LeftComp}(f)$.

(52) If C is connected, then the Go-board of $\mathrm{Cage}(C,n) = \mathrm{Gauge}(C,n)$.

(53) If C is connected, then $\mathrm{N_{min}}(C) \in \mathrm{rightcell}(\mathrm{Cage}(C,n), 1)$.

(54) If C is connected and $i \leq j$, then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)) \subseteq \overline{\mathrm{RightComp}(\mathrm{Cage}(C,i))}$.

(55) If C is connected and $i \leq j$, then $\mathrm{LeftComp}(\mathrm{Cage}(C,i)) \subseteq \mathrm{LeftComp}(\mathrm{Cage}(C,j))$.

(56) If C is connected and $i \leq j$, then $\mathrm{RightComp}(\mathrm{Cage}(C,j)) \subseteq \mathrm{RightComp}(\mathrm{Cage}(C,i))$.

[3] The propositions (45) and (46) have been removed.

7. Preparing the Internal Approximation

Let us consider C, n. The functor X-SpanStart(C,n) yields a natural number and is defined by:

(Def. 2) X-SpanStart$(C,n) = 2^{n -' 1} + 2$.

We now state three propositions:

(57) X-SpanStart$(C,n) =$ Center Gauge(C,n).

(58) $2 <$ X-SpanStart(C,n) and X-SpanStart$(C,n) <$ len Gauge(C,n).

(59) $1 \leq$ X-SpanStart$(C,n) -' 1$ and X-SpanStart$(C,n) -' 1 <$ len Gauge(C,n).

Let us consider C, n. We say that n is sufficiently large for C if and only if:

(Def. 3) There exists j such that $j <$ width Gauge(C,n) and cell(Gauge(C,n), X-SpanStart$(C,n) -' 1, j) \subseteq$ BDD C.

One can prove the following propositions:

(60) If n is sufficiently large for C, then $n \geq 1$.

(61) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) left_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 -' 1, j_1 + 1 \rangle \in$ the indices of Gauge(C,n).

(62) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) left_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 1, j_1 + 1 \rangle \in$ the indices of Gauge(C,n).

(63) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) left_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2 + 1, j_1)$,

(iv) $\langle i_2, j_1 \rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1 -' 1 \rangle \in$ the indices of Gauge(C,n).

(64) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_2+1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2+1)$,

(iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1+1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(65) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1+1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1+1)$.

Then $\langle i_1, j_1+2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(66) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1+1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1+1, j_1)$.

Then $\langle i_1+2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(67) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2+1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2+1, j_1)$,

(iv) $\langle i_2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2 -' 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(68) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_2+1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2+1)$,

(iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1, j_2 -' 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(69) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 + 1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(70) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 1, j_1 -' 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(71) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2 + 1, j_1)$,

(iv) $\langle i_2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(72) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_2 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2 + 1)$,

(iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1 -' 1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

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