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# Gauges and Cages. Part I[1]

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The articles [29], [32], [2], [30], [13], [1], [4], [5], [3], [31], [24], [19], [26], [25], [9], [18], [14], [27], [17], [15], [10], [16], [11], [12], [7], [20], [28], [21], [22], [6], [8], and [23] provide the notation and terminology for this paper.

## 1. Preliminaries

For simplicity, we follow the rules: $i$, $i_1$, $i_2$, $j$, $j_1$, $j_2$, $k$, $m$, $n$, $t$ are natural numbers, $D$ is a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, $E$ is a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$, $C$ is a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$, $G$ is a Go-board, $p$, $q$, $x$ are points of $\mathcal{E}^2_{\mathrm{T}}$, and $r$, $s$ are real numbers.

Next we state several propositions:

(1) For all real numbers $s_1$, $s_3$, $s_4$, $l$ such that $s_1 \leq s_3$ and $s_1 \leq s_4$ and $0 \leq l$ and $l \leq 1$ holds $s_1 \leq (1-l) \cdot s_3 + l \cdot s_4$.

(2) For all real numbers $s_1$, $s_3$, $s_4$, $l$ such that $s_3 \leq s_1$ and $s_4 \leq s_1$ and $0 \leq l$ and $l \leq 1$ holds $(1-l) \cdot s_3 + l \cdot s_4 \leq s_1$.

(3) If $n > 0$, then $m^n \bmod m = 0$.

(4) If $j > 0$ and $i \bmod j = 0$, then $i \div j = \frac{i}{j}$.

(5) If $n > 0$, then $i^n \div i = \frac{i^n}{i}$.

(6) If $0 < n$ and $1 < r$, then $1 < r^n$.

(7) If $r > 1$ and $m > n$, then $r^m > r^n$.

(8) Let $T$ be a non empty topological space, $A$ be a subset of $T$, and $B$, $C$ be subsets of $T$. Suppose $A$ is connected and $C$ is a component of $B$ and $A$ meets $C$ and $A \subseteq B$. Then $A \subseteq C$.

Let $f$ be a finite sequence. The functor $\mathrm{Center}\, f$ yielding a natural number is defined by:

(Def. 1) $\mathrm{Center}\, f = (\mathrm{len}\, f \div 2) + 1$.

The following two propositions are true:

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(9) For every finite sequence $f$ such that $\operatorname{len} f$ is odd holds $\operatorname{len} f = 2 \cdot \operatorname{Center} f - 1$.

(10) For every finite sequence $f$ such that $\operatorname{len} f$ is even holds $\operatorname{len} f = 2 \cdot \operatorname{Center} f - 2$.

## 2. Some Subsets of the Plane

One can verify the following observations:

* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non vertical, non horizontal, and non empty and satisfies conditions of simple closed curve,
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and horizontal, and
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and vertical.

The following propositions are true:

(11) If $p \in \mathrm{N_{most}}(D)$, then $p_{\mathbf{2}} = \text{N-bound}(D)$.

(12) If $p \in \mathrm{E_{most}}(D)$, then $p_{\mathbf{1}} = \text{E-bound}(D)$.

(13) If $p \in \mathrm{S_{most}}(D)$, then $p_{\mathbf{2}} = \text{S-bound}(D)$.

(14) If $p \in \mathrm{W_{most}}(D)$, then $p_{\mathbf{1}} = \text{W-bound}(D)$.

(15) For every subset $D$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{BDD}\, D$ misses $D$.

(16) For every non empty subset $S$ of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve holds $\mathrm{LowerArc}(S) \subseteq S$ and $\mathrm{UpperArc}(S) \subseteq S$.

(17) $p \in \mathrm{VerticalLine}(p_{\mathbf{1}})$.

(18) $[r,s] \in \mathrm{VerticalLine}(r)$.

(19) For every subset $A$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq \mathrm{VerticalLine}(s)$ holds $A$ is vertical.

(20) $\mathrm{proj2}([r,s]) = s$ and $\mathrm{proj1}([r,s]) = r$.

(21) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $r \in [\mathrm{proj2}(p), \mathrm{proj2}(q)]$, then $[p_{\mathbf{1}}, r] \in \mathcal{L}(p,q)$.

(22) If $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $r \in [\mathrm{proj1}(p), \mathrm{proj1}(q)]$, then $[r, p_{\mathbf{2}}] \in \mathcal{L}(p,q)$.

(23) If $p \in \mathrm{VerticalLine}(s)$ and $q \in \mathrm{VerticalLine}(s)$, then $\mathcal{L}(p,q) \subseteq \mathrm{VerticalLine}(s)$.

Let $S$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve. One can verify that $\mathrm{LowerArc}(S)$ is non empty and compact and $\mathrm{UpperArc}(S)$ is non empty and compact.

We now state several propositions:

(24) For all subsets $A$, $B$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $A$ meets $B$ holds $\mathrm{proj2}^\circ A$ meets $\mathrm{proj2}^\circ B$.

(25) For all subsets $A$, $B$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $A$ misses $B$ and $A \subseteq \mathrm{VerticalLine}(s)$ and $B \subseteq \mathrm{VerticalLine}(s)$ holds $\mathrm{proj2}^\circ A$ misses $\mathrm{proj2}^\circ B$.

(26) For every closed subset $S$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $S$ is Bounded holds $\mathrm{proj2}^\circ S$ is closed.

(27) For every subset $S$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $S$ is Bounded holds $\mathrm{proj2}^\circ S$ is bounded.

(28) For every compact subset $S$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj2}^\circ S$ is compact.

In this article we present several logical schemes. The scheme *TRSubsetEx* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset $A$ of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ such that for every point $p$ of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$

for all values of the parameters.

The scheme *TRSubsetUniq* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

Let $A$, $B$ be subsets of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$. Suppose for every point $p$ of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$ and for every point $p$ of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in B$ iff $\mathcal{P}[p]$. Then $A = B$

for all values of the parameters.

Let $p$ be a point of $\mathcal{E}^{2}_{\mathrm{T}}$. The functor NorthHalfline$\,p$ yields a subset of $\mathcal{E}^{2}_{\mathrm{T}}$ and is defined by:

(Def. 2) For every point $x$ of $\mathcal{E}^{2}_{\mathrm{T}}$ holds $x \in$ NorthHalfline$\,p$ iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \geq p_{\mathbf{2}}$.

The functor EastHalfline$\,p$ yielding a subset of $\mathcal{E}^{2}_{\mathrm{T}}$ is defined as follows:

(Def. 3) For every point $x$ of $\mathcal{E}^{2}_{\mathrm{T}}$ holds $x \in$ EastHalfline$\,p$ iff $x_{\mathbf{1}} \geq p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The functor SouthHalfline$\,p$ yielding a subset of $\mathcal{E}^{2}_{\mathrm{T}}$ is defined as follows:

(Def. 4) For every point $x$ of $\mathcal{E}^{2}_{\mathrm{T}}$ holds $x \in$ SouthHalfline$\,p$ iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \leq p_{\mathbf{2}}$.

The functor WestHalfline$\,p$ yields a subset of $\mathcal{E}^{2}_{\mathrm{T}}$ and is defined as follows:

(Def. 5) For every point $x$ of $\mathcal{E}^{2}_{\mathrm{T}}$ holds $x \in$ WestHalfline$\,p$ iff $x_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The following propositions are true:

(29) NorthHalfline$\,p = \{q; q$ ranges over points of $\mathcal{E}^{2}_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \wedge q_{\mathbf{2}} \geq p_{\mathbf{2}}\}$.

(30) NorthHalfline$\,p = \{[p_{\mathbf{1}}, r]; r$ ranges over elements of $\mathbb{R}$: $r \geq p_{\mathbf{2}}\}$.

(31) EastHalfline$\,p = \{q; q$ ranges over points of $\mathcal{E}^{2}_{\mathrm{T}}$: $q_{\mathbf{1}} \geq p_{\mathbf{1}} \wedge q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(32) EastHalfline$\,p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \geq p_{\mathbf{1}}\}$.

(33) SouthHalfline$\,p = \{q; q$ ranges over points of $\mathcal{E}^{2}_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \wedge q_{\mathbf{2}} \leq p_{\mathbf{2}}\}$.

(34) SouthHalfline$\,p = \{[p_{\mathbf{1}}, r]; r$ ranges over elements of $\mathbb{R}$: $r \leq p_{\mathbf{2}}\}$.

(35) WestHalfline$\,p = \{q; q$ ranges over points of $\mathcal{E}^{2}_{\mathrm{T}}$: $q_{\mathbf{1}} \leq p_{\mathbf{1}} \wedge q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(36) WestHalfline$\,p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \leq p_{\mathbf{1}}\}$.

Let $p$ be a point of $\mathcal{E}^{2}_{\mathrm{T}}$. One can verify the following observations:

* NorthHalfline$\,p$ is non empty and convex,
* EastHalfline$\,p$ is non empty and convex,
* SouthHalfline$\,p$ is non empty and convex, and
* WestHalfline$\,p$ is non empty and convex.

## 3. Goboards

The following propositions are true:

(37) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$, then $G \circ (i, j) \in \mathcal{L}(G \circ (i, 1), G \circ (i, \operatorname{width} G))$.

(38) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$, then $G \circ (i, j) \in \mathcal{L}(G \circ (1, j), G \circ (\operatorname{len} G, j))$.

(39) If $1 \leq j_1$ and $j_1 \leq \operatorname{width} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 \leq \operatorname{len} G$, then $(G \circ (i_1, j_1))_{\mathbf{1}} \leq (G \circ (i_2, j_2))_{\mathbf{1}}$.

(40) If $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq i_2$ and $i_2 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 \leq j_2$ and $j_2 \leq \operatorname{width} G$, then $(G \circ (i_1, j_1))_{\mathbf{2}} \leq (G \circ (i_2, j_2))_{\mathbf{2}}$.

(41) Let $f$ be a non constant standard special circular sequence. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leq t$ and $t \leq \operatorname{len} G$. Then $(G \circ (t, \operatorname{width} G))_{\mathbf{2}} \geq \text{N-bound}(\widetilde{\mathcal{L}}(f))$.

(42) Let $f$ be a non constant standard special circular sequence. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leq t$ and $t \leq \operatorname{width} G$. Then $(G \circ (1, t))_{\mathbf{1}} \leq \text{W-bound}(\widetilde{\mathcal{L}}(f))$.

(43) Let $f$ be a non constant standard special circular sequence. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leq t$ and $t \leq \operatorname{len} G$. Then $(G \circ (t, 1))_{\mathbf{2}} \leq \text{S-bound}(\widetilde{\mathcal{L}}(f))$.

(44) Let $f$ be a non constant standard special circular sequence. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leq t$ and $t \leq \operatorname{width} G$. Then $(G \circ (\operatorname{len} G, t))_{\mathbf{1}} \geq \text{E-bound}(\widetilde{\mathcal{L}}(f))$.

(45) If $i \leq \operatorname{len} G$ and $j \leq \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is non empty.

(46) If $i \leq \operatorname{len} G$ and $j \leq \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is connected.

(47) If $i \leq \operatorname{len} G$, then $\operatorname{cell}(G, i, 0)$ is not Bounded.

(48) If $i \leq \operatorname{len} G$, then $\operatorname{cell}(G, i, \operatorname{width} G)$ is not Bounded.

## 4. Gauges

Next we state a number of propositions:

(49) $\operatorname{width} \operatorname{Gauge}(D, n) = 2^n + 3$.

(50) If $i < j$, then $\operatorname{len} \operatorname{Gauge}(D, i) < \operatorname{len} \operatorname{Gauge}(D, j)$.

(51) If $i \leq j$, then $\operatorname{len} \operatorname{Gauge}(D, i) \leq \operatorname{len} \operatorname{Gauge}(D, j)$.

(52) If $m \leq n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{len} \operatorname{Gauge}(D, n)$.

(53) If $m \leq n$ and $1 < i$ and $i < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{width} \operatorname{Gauge}(D, n)$.

(54) Suppose $m \leq n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$ and $1 < j$ and $j < \operatorname{width} \operatorname{Gauge}(D, m)$. Let $i_1$, $j_1$ be natural numbers. If $i_1 = 2^{n -' m} \cdot (i - 2) + 2$ and $j_1 = 2^{n -' m} \cdot (j - 2) + 2$, then $\operatorname{Gauge}(D, m) \circ (i, j) = \operatorname{Gauge}(D, n) \circ (i_1, j_1)$.

(55) If $m \leq n$ and $1 < i$ and $i + 1 < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leq \operatorname{len} \operatorname{Gauge}(D, n)$.

(56) If $m \leq n$ and $1 < i$ and $i + 1 < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leq \operatorname{width} \operatorname{Gauge}(D, n)$.

(57) If $1 \leq i$ and $i \leq \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leq j$ and $j \leq \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $(\operatorname{Gauge}(D, n) \circ (\operatorname{Center} \operatorname{Gauge}(D, n), i))_{\mathbf{1}} = (\operatorname{Gauge}(D, m) \circ (\operatorname{Center} \operatorname{Gauge}(D, m), j))_{\mathbf{1}}$.

(58) If $1 \leq i$ and $i \leq \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leq j$ and $j \leq \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $(\operatorname{Gauge}(D, n) \circ (i, \operatorname{Center} \operatorname{Gauge}(D, n)))_{\mathbf{2}} = (\operatorname{Gauge}(D, m) \circ (j, \operatorname{Center} \operatorname{Gauge}(D, m)))_{\mathbf{2}}$.

(59) If $1 \leq i$ and $i \leq \operatorname{len} \operatorname{Gauge}(C, 1)$, then $(\operatorname{Gauge}(C, 1) \circ (\operatorname{Center} \operatorname{Gauge}(C, 1), i))_{\mathbf{1}} = \frac{\text{W-bound}(C) + \text{E-bound}(C)}{2}$.

(60) If $1 \leq i$ and $i \leq \operatorname{len} \operatorname{Gauge}(C, 1)$, then $(\operatorname{Gauge}(C, 1) \circ (i, \operatorname{Center} \operatorname{Gauge}(C, 1)))_{\mathbf{2}} = \frac{\text{S-bound}(C) + \text{N-bound}(C)}{2}$.

(61) If $1 \leq i$ and $i \leq \operatorname{len}\operatorname{Gauge}(E,n)$ and $1 \leq j$ and $j \leq \operatorname{len}\operatorname{Gauge}(E,m)$ and $m \leq n$, then $(\operatorname{Gauge}(E,n) \circ (i, \operatorname{len}\operatorname{Gauge}(E,n)))_{\mathbf{2}} \leq (\operatorname{Gauge}(E,m) \circ (j, \operatorname{len}\operatorname{Gauge}(E,m)))_{\mathbf{2}}$.

(62) If $1 \leq i$ and $i \leq \operatorname{len}\operatorname{Gauge}(E,n)$ and $1 \leq j$ and $j \leq \operatorname{len}\operatorname{Gauge}(E,m)$ and $m \leq n$, then $(\operatorname{Gauge}(E,n) \circ (\operatorname{len}\operatorname{Gauge}(E,n), i))_{\mathbf{1}} \leq (\operatorname{Gauge}(E,m) \circ (\operatorname{len}\operatorname{Gauge}(E,m), j))_{\mathbf{1}}$.

(63) If $1 \leq i$ and $i \leq \operatorname{len}\operatorname{Gauge}(E,n)$ and $1 \leq j$ and $j \leq \operatorname{len}\operatorname{Gauge}(E,m)$ and $m \leq n$, then $(\operatorname{Gauge}(E,m) \circ (1, j))_{\mathbf{1}} \leq (\operatorname{Gauge}(E,n) \circ (1, i))_{\mathbf{1}}$.

(64) If $1 \leq i$ and $i \leq \operatorname{len}\operatorname{Gauge}(E,n)$ and $1 \leq j$ and $j \leq \operatorname{len}\operatorname{Gauge}(E,m)$ and $m \leq n$, then $(\operatorname{Gauge}(E,m) \circ (j, 1))_{\mathbf{2}} \leq (\operatorname{Gauge}(E,n) \circ (i, 1))_{\mathbf{2}}$.

(65) If $1 \leq m$ and $m \leq n$, then $\mathcal{L}(\operatorname{Gauge}(E,n) \circ (\operatorname{Center}\operatorname{Gauge}(E,n), 1), \operatorname{Gauge}(E,n) \circ (\operatorname{Center}\operatorname{Gauge}(E,n), \operatorname{len}\operatorname{Gauge}(E,n))) \subseteq \mathcal{L}(\operatorname{Gauge}(E,m) \circ (\operatorname{Center}\operatorname{Gauge}(E,m), 1), \operatorname{Gauge}(E,m) \circ (\operatorname{Center}\operatorname{Gauge}(E,m), \operatorname{len}\operatorname{Gauge}(E,m)))$.

(66) If $1 \leq m$ and $m \leq n$ and $1 \leq j$ and $j \leq \operatorname{width}\operatorname{Gauge}(E,n)$, then $\mathcal{L}(\operatorname{Gauge}(E,n) \circ (\operatorname{Center}\operatorname{Gauge}(E,n), 1), \operatorname{Gauge}(E,n) \circ (\operatorname{Center}\operatorname{Gauge}(E,n), j)) \subseteq \mathcal{L}(\operatorname{Gauge}(E,m) \circ (\operatorname{Center}\operatorname{Gauge}(E,m), 1), \operatorname{Gauge}(E,n) \circ$ $(\operatorname{Center}\operatorname{Gauge}(E,n), j))$.

(67) If $1 \leq m$ and $m \leq n$ and $1 \leq j$ and $j \leq \operatorname{width}\operatorname{Gauge}(E,n)$, then $\mathcal{L}(\operatorname{Gauge}(E,m) \circ (\operatorname{Center}\operatorname{Gauge}(E,m), 1), \operatorname{Gauge}(E,n) \circ (\operatorname{Center}\operatorname{Gauge}(E,n), j)) \subseteq \mathcal{L}(\operatorname{Gauge}(E,m) \circ (\operatorname{Center}\operatorname{Gauge}(E,m), 1), \operatorname{Gauge}(E,m) \circ$ $(\operatorname{Center}\operatorname{Gauge}(E,m), \operatorname{len}\operatorname{Gauge}(E,m)))$.

(68) Suppose $m \leq n$ and $1 < i$ and $i+1 < \operatorname{len}\operatorname{Gauge}(E,m)$ and $1 < j$ and $j+1 < \operatorname{width}\operatorname{Gauge}(E,m)$. Let $i_1$, $j_1$ be natural numbers. Suppose $2^{n -' m} \cdot (i-2)+2 \leq i_1$ and $i_1 < 2^{n -' m} \cdot (i-1)+2$ and $2^{n -' m} \cdot (j-2)+2 \leq j_1$ and $j_1 < 2^{n -' m} \cdot (j-1)+2$. Then $\operatorname{cell}(\operatorname{Gauge}(E,n), i_1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E,m), i, j)$.

(69) Suppose $m \leq n$ and $3 \leq i$ and $i < \operatorname{len}\operatorname{Gauge}(E,m)$ and $1 < j$ and $j+1 < \operatorname{width}\operatorname{Gauge}(E,m)$. Let $i_1$, $j_1$ be natural numbers. If $i_1 = 2^{n -' m} \cdot (i-2)+2$ and $j_1 = 2^{n -' m} \cdot (j-2)+2$, then $\operatorname{cell}(\operatorname{Gauge}(E,n), i_1 -' 1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E,m), i -' 1, j)$.

(70) For every compact non vertical non horizontal subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $i \leq \operatorname{len}\operatorname{Gauge}(C,n)$ holds $\operatorname{cell}(\operatorname{Gauge}(C,n), i, 0) \subseteq \operatorname{UBD} C$.

(71) For every compact non vertical non horizontal subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $i \leq \operatorname{len}\operatorname{Gauge}(E,n)$ holds $\operatorname{cell}(\operatorname{Gauge}(E,n), i, \operatorname{width}\operatorname{Gauge}(E,n)) \subseteq \operatorname{UBD} E$.

## 5. Cages

We now state a number of propositions:

(72) If $p \in C$, then NorthHalfline $p$ meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(73) If $p \in C$, then EastHalfline $p$ meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(74) If $p \in C$, then SouthHalfline $p$ meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(75) If $p \in C$, then WestHalfline $p$ meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(76) There exist $k$, $t$ such that $1 \leq k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = \operatorname{Gauge}(C,n) \circ (1,t)$.

(77) There exist $k$, $t$ such that $1 \leq k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{len}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = \operatorname{Gauge}(C,n) \circ (t,1)$.

(78) There exist $k$, $t$ such that $1 \leq k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = \operatorname{Gauge}(C,n) \circ (\operatorname{len}\operatorname{Gauge}(C,n), t)$.

(79) If $1 \leq k$ and $k \leq \operatorname{len}\mathrm{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{len}\mathrm{Gauge}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{Gauge}(C,n) \circ (t, \operatorname{width}\mathrm{Gauge}(C,n))$, then $(\mathrm{Cage}(C,n))_k \in \mathrm{N}_{\mathrm{most}}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(80) If $1 \leq k$ and $k \leq \operatorname{len}\mathrm{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{width}\mathrm{Gauge}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{Gauge}(C,n) \circ (1,t)$, then $(\mathrm{Cage}(C,n))_k \in \mathrm{W}_{\mathrm{most}}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(81) If $1 \leq k$ and $k \leq \operatorname{len}\mathrm{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{len}\mathrm{Gauge}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{Gauge}(C,n) \circ (t,1)$, then $(\mathrm{Cage}(C,n))_k \in \mathrm{S}_{\mathrm{most}}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(82) If $1 \leq k$ and $k \leq \operatorname{len}\mathrm{Cage}(C,n)$ and $1 \leq t$ and $t \leq \operatorname{width}\mathrm{Gauge}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{Gauge}(C,n) \circ (\operatorname{len}\mathrm{Gauge}(C,n),t)$, then $(\mathrm{Cage}(C,n))_k \in \mathrm{E}_{\mathrm{most}}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(83) $\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \text{W-bound}(C) - \frac{\text{E-bound}(C) - \text{W-bound}(C)}{2^n}$.

(84) $\text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \text{S-bound}(C) - \frac{\text{N-bound}(C) - \text{S-bound}(C)}{2^n}$.

(85) $\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \text{E-bound}(C) + \frac{\text{E-bound}(C) - \text{W-bound}(C)}{2^n}$.

(86) $\text{N-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \text{N-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,m))) + \text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,m)))$.

(87) $\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,m))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,m)))$.

(88) If $i < j$, then $\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))) < \text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))$.

(89) If $i < j$, then $\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))) < \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))$.

(90) If $i < j$, then $\text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))) < \text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))$.

(91) If $1 \leq i$ and $i \leq \operatorname{len}\mathrm{Gauge}(C,n)$, then $\text{N-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = (\mathrm{Gauge}(C,n) \circ (i, \operatorname{len}\mathrm{Gauge}(C,n)))_{\mathbf{2}}$.

(92) If $1 \leq i$ and $i \leq \operatorname{len}\mathrm{Gauge}(C,n)$, then $\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = (\mathrm{Gauge}(C,n) \circ (\operatorname{len}\mathrm{Gauge}(C,n), i))_{\mathbf{1}}$.

(93) If $1 \leq i$ and $i \leq \operatorname{len}\mathrm{Gauge}(C,n)$, then $\text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = (\mathrm{Gauge}(C,n) \circ (i,1))_{\mathbf{2}}$.

(94) If $1 \leq i$ and $i \leq \operatorname{len}\mathrm{Gauge}(C,n)$, then $\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = (\mathrm{Gauge}(C,n) \circ (1,i))_{\mathbf{1}}$.

(95) If $x \in C$ and $p \in \mathrm{NorthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{2}} > x_{\mathbf{2}}$.

(96) If $x \in C$ and $p \in \mathrm{EastHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{1}} > x_{\mathbf{1}}$.

(97) If $x \in C$ and $p \in \mathrm{SouthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{2}} < x_{\mathbf{2}}$.

(98) If $x \in C$ and $p \in \mathrm{WestHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{1}} < x_{\mathbf{1}}$.

(99) If $x \in \mathrm{N}_{\mathrm{most}}(C)$ and $p \in \mathrm{NorthHalfline}\, x$ and $1 \leq i$ and $i < \operatorname{len}\mathrm{Cage}(C,n)$ and $p \in \mathcal{L}(\mathrm{Cage}(C,n),i)$, then $\mathcal{L}(\mathrm{Cage}(C,n),i)$ is horizontal.

(100) If $x \in \mathrm{E}_{\mathrm{most}}(C)$ and $p \in \mathrm{EastHalfline}\, x$ and $1 \leq i$ and $i < \operatorname{len}\mathrm{Cage}(C,n)$ and $p \in \mathcal{L}(\mathrm{Cage}(C,n),i)$, then $\mathcal{L}(\mathrm{Cage}(C,n),i)$ is vertical.

(101) If $x \in \mathrm{S}_{\mathrm{most}}(C)$ and $p \in \mathrm{SouthHalfline}\, x$ and $1 \leq i$ and $i < \operatorname{len}\mathrm{Cage}(C,n)$ and $p \in \mathcal{L}(\mathrm{Cage}(C,n),i)$, then $\mathcal{L}(\mathrm{Cage}(C,n),i)$ is horizontal.

(102) If $x \in \mathrm{W}_{\mathrm{most}}(C)$ and $p \in \mathrm{WestHalfline}\, x$ and $1 \leq i$ and $i < \operatorname{len}\mathrm{Cage}(C,n)$ and $p \in \mathcal{L}(\mathrm{Cage}(C,n),i)$, then $\mathcal{L}(\mathrm{Cage}(C,n),i)$ is vertical.

(103) If $x \in \mathrm{N}_{\mathrm{most}}(C)$ and $p \in \mathrm{NorthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(104) If $x \in \mathrm{E}_{\mathrm{most}}(C)$ and $p \in \mathrm{EastHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(105) If $x \in \mathrm{S}_{\mathrm{most}}(C)$ and $p \in \mathrm{SouthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(106) If $x \in \mathrm{W}_{\mathrm{most}}(C)$ and $p \in \mathrm{WestHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $p_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(107) If $x \in \mathrm{N}_{\mathrm{most}}(C)$, then there exists a point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{NorthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{p\}$.

(108) If $x \in \mathrm{E}_{\mathrm{most}}(C)$, then there exists a point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{EastHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{p\}$.

(109) If $x \in \mathrm{S}_{\mathrm{most}}(C)$, then there exists a point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{SouthHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{p\}$.

(110) If $x \in \mathrm{W}_{\mathrm{most}}(C)$, then there exists a point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{WestHalfline}\, x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{p\}$.

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