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On the Decomposition of a Simple Closed Curve into Two Arcs

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Summary. The purpose of the paper is to prove lemmas needed for the Jordan curve theorem. The main result is that the decomposition of a simple closed curve into two arcs with the ends p_1, p_2 is unique in the sense that every arc on the curve with the same ends must be equal to one of them.

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The articles [25], [24], [27], [1], [26], [28], [3], [5], [9], [4], [22], [17], [21], [8], [7], [20], [2], [23], [15], [10], [6], [11], [19], [18], [12], [14], [13], and [16] provide the notation and terminology for this paper.

One can prove the following proposition

(1) Let S_1 be a finite non empty subset of $\mathbb{R}$ and e be a real number. If for every real number r such that $r \in S_1$ holds $r < e$, then $\max S_1 < e$.

For simplicity, we follow the rules: C is a simple closed curve, A, A_1, A_2 are subsets of $\mathcal{E}^2_{\mathrm{T}}$, p, p_1, p_2, q, q_1, q_2 are points of $\mathcal{E}^2_{\mathrm{T}}$, and n is a natural number.

Let us consider n. Observe that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is trivial.

One can prove the following propositions:

(2) For all sets a, b, c, X such that $a \in X$ and $b \in X$ and $c \in X$ holds $\{a,b,c\} \subseteq X$.

(3) $\emptyset_{\mathcal{E}^n_{\mathrm{T}}}$ is Bounded.

(4) $\mathrm{LowerArc}(C) \neq \mathrm{UpperArc}(C)$.

(5) $\mathrm{Segment}(A, p_1, p_2, q_1, q_2) \subseteq A$.

(6) For every non empty topological space T and for all subsets A, B of T such that $A \subseteq B$ holds $T \restriction A$ is a subspace of $T \restriction B$.

(7) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{LSegment}(A, p_1, p_2, q)$.

(8) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{RSegment}(A, p_1, p_2, q)$.

(9) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $q_1 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ and $q_2 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$.

(10) $\mathrm{Segment}(p, q, C) \subseteq C$.

(11) If $p \in C$ and $q \in C$, then $p \leq_C q$ or $q \leq_C p$.

(12) Let X, Y be non empty topological spaces, Y_0 be a non empty subspace of Y, f be a map from X into Y, and g be a map from X into Y_0. If $f = g$ and f is continuous, then g is continuous.

(13) Let S, T be non empty topological spaces, S_0 be a non empty subspace of S, T_0 be a non empty subspace of T, and f be a map from S into T. Suppose f is a homeomorphism. Let g be a map from S_0 into T_0. If $g = f{\restriction}S_0$ and g is onto, then g is a homeomorphism.

(14) Let P_1, P_2, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and $P_2 \cap P_3 = \{p_1, p_2\}$ and $P_1 \subseteq P_2 \cup P_3$. Then $P_1 = P_2$ or $P_1 = P_3$.

(15) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$. Then $A_1 \cup A_2 = C$ and $A_1 \cap A_2 = \{p_1, p_2\}$.

(16) Let A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If A_1 is an arc from p_1 to p_2 and $A_1 \cap A_2 = \{q_1, q_2\}$, then $A_1 \neq A_2$.

(17) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \cap A_2 = \{p_1, p_2\}$. Then $A_1 \cup A_2 = C$.

(18) Suppose $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$ and A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2. Let given A. If A is an arc from p_1 to p_2 and $A \subseteq C$, then $A = A_1$ or $A = A_2$.

(19) Let C be a simple closed curve and A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If A is an arc from $\mathrm{W}_{\mathrm{min}}(C)$ to $\mathrm{E}_{\mathrm{max}}(C)$ and $A \subseteq C$, then $A = \mathrm{LowerArc}(C)$ or $A = \mathrm{UpperArc}(C)$.

(20) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}){\restriction}A$ and there exist real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leq s_1$ and $s_1 \leq s_2$ and $s_2 \leq 1$.

(21) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}){\restriction}A$ and there exist real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leq s_1$ and $s_1 < s_2$ and $s_2 \leq 1$.

(22) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is non empty.

(23) If $p \in C$, then $p \in \mathrm{Segment}(p, \mathrm{W}_{\mathrm{min}}(C), C)$ and $\mathrm{W}_{\mathrm{min}}(C) \in \mathrm{Segment}(p, \mathrm{W}_{\mathrm{min}}(C), C)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is continuous if and only if:

(Def. 1) f is continuous on $\mathrm{dom}\, f$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Let us observe that f is continuous if and only if:

(Def. 2) f is continuous on $\mathbb{R}$.

Let a, b be real numbers. The functor $\mathrm{AffineMap}(a, b)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 3) For every real number x holds $(\mathrm{AffineMap}(a, b))(x) = a \cdot x + b$.

Let a, b be real numbers. One can check that $\mathrm{AffineMap}(a, b)$ is continuous.
Let us observe that there exists a function from $\mathbb{R}$ into $\mathbb{R}$ which is continuous.
Next we state a number of propositions:

(24) Let f, g be continuous partial functions from $\mathbb{R}$ to $\mathbb{R}$. Then $g \cdot f$ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$.

(25) For all real numbers a, b holds $(\mathrm{AffineMap}(a,b))(0) = b$.

(26) For all real numbers a, b holds $(\mathrm{AffineMap}(a,b))(1) = a+b$.

(27) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{AffineMap}(a,b)$ is one-to-one.

(28) For all real numbers a, b, x, y such that $a > 0$ and $x < y$ holds $(\mathrm{AffineMap}(a,b))(x) < (\mathrm{AffineMap}(a,b))(y)$.

(29) For all real numbers a, b, x, y such that $a < 0$ and $x < y$ holds $(\mathrm{AffineMap}(a,b))(x) > (\mathrm{AffineMap}(a,b))(y)$.

(30) For all real numbers a, b, x, y such that $a \geq 0$ and $x \leq y$ holds $(\mathrm{AffineMap}(a,b))(x) \leq (\mathrm{AffineMap}(a,b))(y)$.

(31) For all real numbers a, b, x, y such that $a \leq 0$ and $x \leq y$ holds $(\mathrm{AffineMap}(a,b))(x) \geq (\mathrm{AffineMap}(a,b))(y)$.

(32) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{rng}\,\mathrm{AffineMap}(a,b) = \mathbb{R}$.

(33) For all real numbers a, b such that $a \neq 0$ holds $(\mathrm{AffineMap}(a,b))^{-1} = \mathrm{AffineMap}(a^{-1}, -\frac{b}{a})$.

(34) For all real numbers a, b such that $a > 0$ holds $(\mathrm{AffineMap}(a,b))^{\circ}[0,1] = [b, a+b]$.

(35) For every map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \neq 0$ and $f = \mathrm{AffineMap}(a,b)$ holds f is a homeomorphism.

(36) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is an arc from q_1 to q_2.

(37) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P \subseteq C$ and P is an arc from p_1 to p_2 and $\mathrm{W}_{\min}(C) \in P$ and $\mathrm{E}_{\max}(C) \in P$. Then $\mathrm{UpperArc}(C) \subseteq P$ or $\mathrm{LowerArc}(C) \subseteq P$.

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