JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

Some Lemmas for the Jordan Curve Theorem[1]

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Summary. I present some miscellaneous simple facts that are still missing in the library. The only common feature is that, most of them, were needed as lemmas in the proof of the Jordan curve theorem.

MML Identifier: JCT_MISC.

WWW: http://mizar.org/JFM/Vol12/jct_misc.html

The articles [18], [6], [23], [19], [24], [4], [25], [5], [3], [2], [21], [8], [1], [16], [13], [22], [12], [10], [9], [17], [15], [20], [7], [11], and [14] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *NonEmpty* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{A}\}$ is non empty

for all values of the parameters.

One can prove the following propositions:

(3)[1] For all sets A, B and for every function f such that $A \subseteq \operatorname{dom} f$ and $f^\circ A \subseteq B$ holds $A \subseteq f^{-1}(B)$.

(4) For every function f and for all sets A, B such that A misses B holds $f^{-1}(A)$ misses $f^{-1}(B)$.

(5) Let S, X be sets, f be a function from S into X, and A be a subset of X such that if $X = \emptyset$, then $S = \emptyset$. Then $(f^{-1}(A))^{\rm c} = f^{-1}(A^{\rm c})$.

(6) Let S be a 1-sorted structure, X be a non empty set, f be a function from the carrier of S into X, and A be a subset of X. Then $(f^{-1}(A))^{\rm c} = f^{-1}(A^{\rm c})$.

We adopt the following rules: i, j, m, n denote natural numbers and r, s, r_0, s_0, t denote real numbers.

Next we state several propositions:

(7) If $m \leq n$, then $n -' (n -' m) = m$.

(9)[2] For all real numbers a, b such that $r \in [a,b]$ and $s \in [a,b]$ holds $\frac{r+s}{2} \in [a,b]$.

(10) For every increasing sequence N_1 of naturals and for all i, j such that $i \leq j$ holds $N_1(i) \leq N_1(j)$.

[1] This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

[1] The propositions (1) and (2) have been removed.

[2] The proposition (8) has been removed.

(11) $||r_0 - s_0| - |r - s|| \leq |r_0 - r| + |s_0 - s|$.

(12) If $t \in]r,s[$, then $|t| < \max(|r|,|s|)$.

Let A, B, C be non empty sets and let f be a function from A into $[:B,C:]$. Then $\mathrm{pr1}(f)$ is a function from A into B and it can be characterized by the condition:

(Def. 1) For every element x of A holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

Then $\mathrm{pr2}(f)$ is a function from A into C and it can be characterized by the condition:

(Def. 2) For every element x of A holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

The scheme *DoubleChoice* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function a from $\mathcal{A}$ into $\mathcal{B}$ and there exists a function b from $\mathcal{A}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ holds $\mathcal{P}[i,a(i),b(i)]$

provided the parameters satisfy the following condition:

- For every element i of $\mathcal{A}$ there exists an element a_1 of $\mathcal{B}$ and there exists an element b_1 of $\mathcal{C}$ such that $\mathcal{P}[i,a_1,b_1]$.

Next we state the proposition

(13) Let S, T be non empty topological spaces and G be a subset of $[:S,T:]$. Suppose that for every point x of $[:S,T:]$ such that $x \in G$ there exists a subset G_1 of S and there exists a subset G_2 of T such that G_1 is open and G_2 is open and $x \in [:G_1,G_2:]$ and $[:G_1,G_2:] \subseteq G$. Then G is open.

2. Topological Properties of Sets of Real Numbers

One can prove the following proposition

(14) For all compact subsets A, B of $\mathbb{R}$ holds $A \cap B$ is compact.

Let A be a subset of $\mathbb{R}$. We say that A is connected if and only if:

(Def. 3) For all real numbers r, s such that $r \in A$ and $s \in A$ holds $[r,s] \subseteq A$.

Next we state the proposition

(15) Let T be a non empty topological space, f be a continuous real map of T, and A be a subset of T. If A is connected, then $f^{\circ}A$ is connected.

Let A, B be subsets of $\mathbb{R}$. The functor $\rho(A,B)$ yielding a real number is defined by:

(Def. 4) There exists a subset X of $\mathbb{R}$ such that $X = \{|r-s|; r$ ranges over elements of $\mathbb{R}$, s ranges over elements of $\mathbb{R}$: $r \in A \ \wedge \ s \in B\}$ and $\rho(A,B) = \inf X$.

Let us observe that the functor $\rho(A,B)$ is commutative.

We now state several propositions:

(16) For all subsets A, B of $\mathbb{R}$ and for all r, s such that $r \in A$ and $s \in B$ holds $|r-s| \geq \rho(A,B)$.

(17) For all subsets A, B of $\mathbb{R}$ and for all non empty subsets C, D of $\mathbb{R}$ such that $C \subseteq A$ and $D \subseteq B$ holds $\rho(A,B) \leq \rho(C,D)$.

(18) For all non empty compact subsets A, B of $\mathbb{R}$ there exist real numbers r, s such that $r \in A$ and $s \in B$ and $\rho(A,B) = |r-s|$.

(19) For all non empty compact subsets A, B of $\mathbb{R}$ holds $\rho(A,B) \geq 0$.

(20) For all non empty compact subsets A, B of $\mathbb{R}$ such that A misses B holds $\rho(A,B) > 0$.

(21) Let e, f be real numbers and A, B be compact subsets of $\mathbb{R}$. Suppose A misses B and $A \subseteq [e,f]$ and $B \subseteq [e,f]$. Let S be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. Suppose that for every natural number i holds $S(i)$ is connected and $S(i)$ meets A and $S(i)$ meets B. Then there exists a real number r such that $r \in [e,f]$ and $r \notin A \cup B$ and for every natural number i there exists a natural number k such that $i \leq k$ and $r \in S(k)$.

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Received August 28, 2000

Published January 2, 2004