JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Some Isomorphisms Between Functor Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. We define some well known isomorphisms between functor categories: between $A^{\circlearrowright(o,m)}$ and A, between $C^{[:A,B:]}$ and $(C^B)^A$, and between $[:B,C:]^A$ and $[:B^A,C^A:]$. Compare [9] and [8]. Unfortunately in this paper "functor" is used in two different meanings, as a lingual function and as a functor between categories.

MML Identifier: ISOCAT_2.

WWW: http://mizar.org/JFM/Vol4/isocat_2.html

The articles [10], [6], [14], [15], [16], [3], [4], [2], [1], [11], [5], [7], [13], and [12] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let A, B, C be non empty sets and let f be a function from A into C^B. Then uncurry f is a function from $[:A, B:]$ into C.

We now state several propositions:

(1) For all non empty sets A, B, C and for every function f from A into C^B holds curry uncurry $f = f$.

(2) Let A, B, C be non empty sets, f be a function from A into C^B, a be an element of A, and b be an element of B. Then $(\text{uncurry} f)(\langle a, b\rangle) = f(a)(b)$.

(3) For every set x and for every non empty set A and for all functions f, g from $\{x\}$ into A such that $f(x) = g(x)$ holds $f = g$.

(4) For all non empty sets A, B and for every element x of A and for every function f from A into B holds $f(x) \in \text{rng} f$.

(5) Let A, B, C be non empty sets and f, g be functions from A into $[:B, C:]$. If $\pi_1(B\times C)\cdot f = \pi_1(B\times C)\cdot g$ and $\pi_2(B\times C)\cdot f = \pi_2(B\times C)\cdot g$, then $f = g$.

In the sequel A, B, C denote categories.

We now state two propositions:

(6) For every morphism f of A holds $\text{id}_{\text{cod} f}\cdot f = f$.

(7) For every morphism f of A holds $f\cdot \text{id}_{\text{dom} f} = f$.

In the sequel o, m are sets.

The following propositions are true:

(8) o is an object of B^A iff o is a functor from A to B.

(9) Let f be a morphism of B^A. Then there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 and $\mathrm{dom}\, f = F_1$ and $\mathrm{cod}\, f = F_2$ and $f = \langle\langle F_1, F_2\rangle, t\rangle$.

2. The isomorphism between $A^{\dot{\circlearrowleft}(o,m)}$ and A

Let us consider A, B and let a be an object of A. The functor $a \mapsto B$ yields a functor from B^A to B and is defined by the condition (Def. 1).

(Def. 1) Let F_1, F_2 be functors from A to B and t be a natural transformation from F_1 to F_2. If F_1 is naturally transformable to F_2, then $(a \mapsto B)(\langle\langle F_1, F_2\rangle, t\rangle) = t(a)$.

The following proposition is true

(11)[1] $A^{\dot{\circlearrowleft}(o,m)} \cong A$.

3. The isomorphism between $C^{[:A,B:]}$ and $(C^B)^A$

One can prove the following four propositions:

(12) For every functor F from $[:A, B:]$ to C and for every object a of A and for every object b of B holds $F(a, -)(b) = F(\langle a, b\rangle)$.

(13) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B holds $\hom(a_1, a_2) \neq \emptyset$ and $\hom(b_1, b_2) \neq \emptyset$ iff $\hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$.

(14) Let a_1, a_2 be objects of A and b_1, b_2 be objects of B. Suppose $\hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$. Let f be a morphism of A and g be a morphism of B. Then $\langle f, g\rangle$ is a morphism from $\langle a_1, b_1\rangle$ to $\langle a_2, b_2\rangle$ if and only if f is a morphism from a_1 to a_2 and g is a morphism from b_1 to b_2.

(15) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2 and a be an object of A. Then $F_1(a, -)$ is naturally transformable to $F_2(a, -)$ and $(\mathrm{curry}\, t)(a)$ is a natural transformation from $F_1(a, -)$ to $F_2(a, -)$.

Let us consider A, B, C, let F be a functor from $[:A, B:]$ to C, and let f be a morphism of A. The functor $\mathrm{curry}(F, f)$ yields a function from the morphisms of B into the morphisms of C and is defined as follows:

(Def. 2) $\mathrm{curry}(F, f) = (\mathrm{curry}\, F)(f)$.

We now state two propositions:

(16) Let a_1, a_2 be objects of A, b_1, b_2 be objects of B, f be a morphism of A, and g be a morphism of B. If $f \in \hom(a_1, a_2)$ and $g \in \hom(b_1, b_2)$, then $\langle f, g\rangle \in \hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle)$.

(17) Let F be a functor from $[:A, B:]$ to C and a, b be objects of A. Suppose $\hom(a, b) \neq \emptyset$. Let f be a morphism from a to b. Then

(i) $F(a, -)$ is naturally transformable to $F(b, -)$, and

(ii) $\mathrm{curry}(F, f) \cdot$ the id-map of B is a natural transformation from $F(a, -)$ to $F(b, -)$.

Let us consider A, B, C, let F be a functor from $[:A, B:]$ to C, and let f be a morphism of A. The functor $F(f, -)$ yielding a natural transformation from $F(\mathrm{dom}\, f, -)$ to $F(\mathrm{cod}\, f, -)$ is defined by:

(Def. 3) $F(f, -) = \mathrm{curry}(F, f) \cdot$ the id-map of B.

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[1] The proposition (10) has been removed.

One can prove the following propositions:

(18) For every functor F from $[:A, B:]$ to C and for every morphism g of A holds $F(\operatorname{dom} g, -)$ is naturally transformable to $F(\operatorname{cod} g, -)$.

(19) For every functor F from $[:A, B:]$ to C and for every morphism f of A and for every object b of B holds $F(f, -)(b) = F(\langle f, \mathrm{id}_b \rangle)$.

(20) For every functor F from $[:A, B:]$ to C and for every object a of A holds $\mathrm{id}_{F(a,-)} = F(\mathrm{id}_a, -)$.

(21) Let F be a functor from $[:A, B:]$ to C and g, f be morphisms of A. Suppose $\operatorname{dom} g = \operatorname{cod} f$. Let t be a natural transformation from $F(\operatorname{dom} f, -)$ to $F(\operatorname{dom} g, -)$. If $t = F(f, -)$, then $F(g \cdot f, -) = F(g, -) \circ t$.

Let us consider A, B, C and let F be a functor from $[:A, B:]$ to C. The functor $\operatorname{export}(F)$ yields a functor from A to C^B and is defined as follows:

(Def. 4) For every morphism f of A holds $(\operatorname{export}(F))(f) = \langle \langle F(\operatorname{dom} f, -), F(\operatorname{cod} f, -) \rangle, F(f, -) \rangle$.

We now state four propositions:

(24)[2] For every functor F from $[:A, B:]$ to C and for every object a of A holds $(\operatorname{export}(F))(a) = F(a, -)$.

(25) For every functor F from $[:A, B:]$ to C and for every object a of A holds $(\operatorname{export}(F))(a)$ is a functor from B to C.

(26) For all functors F_1, F_2 from $[:A, B:]$ to C such that $\operatorname{export}(F_1) = \operatorname{export}(F_2)$ holds $F_1 = F_2$.

(27) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then

(i) $\operatorname{export}(F_1)$ is naturally transformable to $\operatorname{export}(F_2)$, and

(ii) there exists a natural transformation G from $\operatorname{export}(F_1)$ to $\operatorname{export}(F_2)$ such that for every function s from $[:$the objects of A, the objects of $B:]$ into the morphisms of C such that $t = s$ and for every object a of A holds $G(a) = \langle \langle (\operatorname{export}(F_1))(a), (\operatorname{export}(F_2))(a) \rangle, (\operatorname{curry} s)(a) \rangle$.

Let us consider A, B, C and let F_1, F_2 be functors from $[:A, B:]$ to C. Let us assume that F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. The functor $\operatorname{export}(t)$ yielding a natural transformation from $\operatorname{export}(F_1)$ to $\operatorname{export}(F_2)$ is defined by the condition (Def. 5).

(Def. 5) Let s be a function from $[:$the objects of A, the objects of $B:]$ into the morphisms of C. If $t = s$, then for every object a of A holds $(\operatorname{export}(t))(a) = \langle \langle (\operatorname{export}(F_1))(a), (\operatorname{export}(F_2))(a) \rangle, (\operatorname{curry} s)(a) \rangle$.

We now state several propositions:

(28) For every functor F from $[:A, B:]$ to C holds $\mathrm{id}_{\operatorname{export}(F)} = \operatorname{export}(\mathrm{id}_F)$.

(29) Let F_1, F_2, F_3 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3. Let t_1 be a natural transformation from F_1 to F_2 and t_2 be a natural transformation from F_2 to F_3. Then $\operatorname{export}(t_2 \circ t_1) = \operatorname{export}(t_2) \circ \operatorname{export}(t_1)$.

(30) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t_1, t_2 be natural transformations from F_1 to F_2. If $\operatorname{export}(t_1) = \operatorname{export}(t_2)$, then $t_1 = t_2$.

(31) For every functor G from A to C^B there exists a functor F from $[:A, B:]$ to C such that $G = \operatorname{export}(F)$.

[2] The propositions (22) and (23) have been removed.

(32) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose export(F_1) is naturally transformable to export(F_2). Let t be a natural transformation from export(F_1) to export(F_2). Then F_1 is naturally transformable to F_2 and there exists a natural transformation u from F_1 to F_2 such that $t = \mathrm{export}(u)$.

Let us consider A, B, C. The functor $\mathbf{export}_{A,B,C}$ yielding a functor from $C^{[:A,B:]}$ to $(C^B)^A$ is defined by the condition (Def. 6).

(Def. 6) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathbf{export}_{A,B,C}(\langle\langle F_1, F_2\rangle, t\rangle) = \langle\langle \mathrm{export}(F_1), \mathrm{export}(F_2)\rangle, \mathrm{export}(t)\rangle$.

Next we state two propositions:

(33) $\mathbf{export}_{A,B,C}$ is an isomorphism.

(34) $C^{[:A,B:]} \cong (C^B)^A$.

4. The isomorphism between $[:B,C:]^A$ and $[:B^A,C^A:]$

We now state the proposition

(35) Let F_1, F_2 be functors from A to B and G be a functor from B to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $G \cdot t = G \cdot (t$ **qua** function).

Let us consider A, B. Then $\pi_1(A \times B)$ is a functor from $[:A, B:]$ to A. Then $\pi_2(A \times B)$ is a functor from $[:A, B:]$ to B.

Let us consider A, B, C, let F be a functor from A to B, and let G be a functor from A to C. Then $\langle F, G\rangle$ is a functor from A to $[:B, C:]$.

Let us consider A, B, C and let F be a functor from A to $[:B, C:]$. The functor $\pi_1 \cdot F$ yields a functor from A to B and is defined by:

(Def. 7) $\pi_1 \cdot F = \pi_1(B \times C) \cdot F$.

The functor $\pi_2 \cdot F$ yields a functor from A to C and is defined by:

(Def. 8) $\pi_2 \cdot F = \pi_2(B \times C) \cdot F$.

Next we state two propositions:

(36) For every functor F from A to B and for every functor G from A to C holds $\pi_1 \cdot \langle F, G\rangle = F$ and $\pi_2 \cdot \langle F, G\rangle = G$.

(37) For all functors F, G from A to $[:B, C:]$ such that $\pi_1 \cdot F = \pi_1 \cdot G$ and $\pi_2 \cdot F = \pi_2 \cdot G$ holds $F = G$.

Let us consider A, B, C, let F_1, F_2 be functors from A to $[:B, C:]$, and let t be a natural transformation from F_1 to F_2. The functor $\pi_1 \cdot t$ yielding a natural transformation from $\pi_1 \cdot F_1$ to $\pi_1 \cdot F_2$ is defined as follows:

(Def. 9) $\pi_1 \cdot t = \pi_1(B \times C) \cdot t$.

The functor $\pi_2 \cdot t$ yields a natural transformation from $\pi_2 \cdot F_1$ to $\pi_2 \cdot F_2$ and is defined as follows:

(Def. 10) $\pi_2 \cdot t = \pi_2(B \times C) \cdot t$.

Next we state several propositions:

(38) Let F, G be functors from A to $[:B, C:]$. Suppose F is naturally transformable to G. Then $\pi_1 \cdot F$ is naturally transformable to $\pi_1 \cdot G$ and $\pi_2 \cdot F$ is naturally transformable to $\pi_2 \cdot G$.

(39) Let F_1, F_2, G_1, G_2 be functors from A to $[:B,C:]$. Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2. Let s be a natural transformation from F_1 to F_2 and t be a natural transformation from G_1 to G_2. If $\pi_1 \cdot s = \pi_1 \cdot t$ and $\pi_2 \cdot s = \pi_2 \cdot t$, then $s = t$.

(40) For every functor F from A to $[:B,C:]$ holds $\mathrm{id}_{\pi_1 \cdot F} = \pi_1 \cdot (\mathrm{id}_F)$ and $\mathrm{id}_{\pi_2 \cdot F} = \pi_2 \cdot (\mathrm{id}_F)$.

(41) Let F, G, H be functors from A to $[:B,C:]$. Suppose F is naturally transformable to G and G is naturally transformable to H. Let s be a natural transformation from F to G and t be a natural transformation from G to H. Then $\pi_1 \cdot (t \circ s) = \pi_1 \cdot t \circ \pi_1 \cdot s$ and $\pi_2 \cdot (t \circ s) = \pi_2 \cdot t \circ \pi_2 \cdot s$.

(42) Let F be a functor from A to B, G be a functor from A to C, and a, b be objects of A. If $\hom(a,b) \neq \emptyset$, then for every morphism f from a to b holds $\langle F,G \rangle(f) = \langle\!\langle F(f), G(f) \rangle\!\rangle$.

(43) Let F be a functor from A to B, G be a functor from A to C, and a be an object of A. Then $\langle F,G \rangle(a) = \langle\!\langle F(a), G(a) \rangle\!\rangle$.

(44) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is transformable to G_1 and F_2 is transformable to G_2. Then $\langle F_1,F_2 \rangle$ is transformable to $\langle G_1,G_2 \rangle$.

Let us consider A, B, C, let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C. Let us assume that F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1 and let t_2 be a transformation from F_2 to G_2. The functor $\langle t_1,t_2 \rangle$ yields a transformation from $\langle F_1,F_2 \rangle$ to $\langle G_1,G_2 \rangle$ and is defined as follows:

(Def. 11) $\langle t_1,t_2 \rangle = \langle t_1,t_2 \rangle$.

Next we state two propositions:

(45) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1, t_2 be a transformation from F_2 to G_2, and a be an object of A. Then $\langle t_1,t_2 \rangle(a) = \langle\!\langle t_1(a), t_2(a) \rangle\!\rangle$.

(46) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Then $\langle F_1,F_2 \rangle$ is naturally transformable to $\langle G_1,G_2 \rangle$.

Let us consider A, B, C, let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C. Let us assume that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1 and let t_2 be a natural transformation from F_2 to G_2. The functor $\langle t_1,t_2 \rangle$ yields a natural transformation from $\langle F_1,F_2 \rangle$ to $\langle G_1,G_2 \rangle$ and is defined as follows:

(Def. 12) $\langle t_1,t_2 \rangle = \langle t_1,t_2 \rangle$.

One can prove the following proposition

(47) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1 and t_2 be a natural transformation from F_2 to G_2. Then $\pi_1 \cdot \langle t_1,t_2 \rangle = t_1$ and $\pi_2 \cdot \langle t_1,t_2 \rangle = t_2$.

Let us consider A, B, C. The functor $\mathbf{distribute}_{A,B,C}$ yields a functor from $[:B,C:]^A$ to $[:B^A, C^A:]$ and is defined by the condition (Def. 13).

(Def. 13) Let F_1, F_2 be functors from A to $[:B,C:]$. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathbf{distribute}_{A,B,C}(\langle\!\langle \langle\!\langle F_1, F_2 \rangle\!\rangle, t \rangle\!\rangle) = \langle\!\langle \langle\!\langle \langle\!\langle \pi_1 \cdot F_1, \pi_1 \cdot F_2 \rangle\!\rangle, \pi_1 \cdot t \rangle\!\rangle, \langle\!\langle \langle\!\langle \pi_2 \cdot F_1, \pi_2 \cdot F_2 \rangle\!\rangle, \pi_2 \cdot t \rangle\!\rangle \rangle\!\rangle$.

One can prove the following two propositions:

(48) $\mathbf{distribute}_{A,B,C}$ is an isomorphism.

(49) $[:B,C:]^A \cong [:B^A, C^A:]$.

REFERENCES

[1] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_5.html`.

[2] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_3.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Introduction to categories and functors. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/cat_1.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Czesław Byliński. Subcategories and products of categories. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/cat_2.html`.

[8] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Heidelberg, Berlin, 1971.

[9] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*, volume 45 of *Biblioteka Matematyczna*. PWN, Warszawa, 1978.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[11] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fraenkel.html`.

[12] Andrzej Trybulec. Isomorphisms of categories. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/isocat_1.html`.

[13] Andrzej Trybulec. Natural transformations. Discrete categories. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/nattra_1.html`.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[16] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received June 5, 1992

Published January 2, 2004