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The Definition of the Riemann Definite Integral and some Related Lemmas

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Summary. This article introduces the Riemann definite integral on the closed interval of real. We present the definitions and related lemmas of the closed interval. We formalize the concept of the Riemann definite integral and the division of the closed interval of real, and prove the additivity of the integral.

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The articles [26], [7], [29], [2], [27], [14], [4], [30], [16], [15], [20], [24], [10], [12], [3], [25], [21], [8], [28], [17], [18], [23], [9], [11], [19], [22], [1], [6], [13], and [5] provide the notation and terminology for this paper.

1. Definition of Closed Interval and its Properties

For simplicity, we adopt the following rules: a, b denote real numbers, F, G, H denote finite sequences of elements of $\mathbb{R}$, i, j, k denote natural numbers, X denotes a non empty set, and x_1 denotes a set.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is closed-interval if and only if:

(Def. 1) There exist real numbers a, b such that $a \leq b$ and $I_1 = [a,b]$.

Let us note that there exists a subset of $\mathbb{R}$ which is closed-interval.

In the sequel A denotes a closed-interval subset of $\mathbb{R}$.

Next we state two propositions:

(1) A is compact.

(2) A is non empty.

One can verify that every subset of $\mathbb{R}$ which is closed-interval is also non empty and compact.

Next we state the proposition

(3) A is lower bounded and upper bounded.

Let us observe that every subset of $\mathbb{R}$ which is closed-interval is also bounded.

Let us note that there exists a subset of $\mathbb{R}$ which is closed-interval.

In the sequel A denotes a closed-interval subset of $\mathbb{R}$.

The following propositions are true:

(4) There exist a, b such that $a \leq b$ and $a = \inf A$ and $b = \sup A$.

(5) $A = [\inf A, \sup A]$.

(6) For all real numbers a_1, a_2, b_1, b_2 such that $A = [a_1, b_1]$ and $A = [a_2, b_2]$ holds $a_1 = a_2$ and $b_1 = b_2$.

2. DEFINITION OF DIVISION OF CLOSED INTERVAL AND ITS PROPERTIES

Let A be a non empty compact subset of $\mathbb{R}$. A non empty increasing finite sequence of elements of $\mathbb{R}$ is said to be a DivisionPoint of A if:

(Def. 2) $\operatorname{rng} \mathrm{it} \subseteq A$ and $\mathrm{it}(\operatorname{len} \mathrm{it}) = \sup A$.

Let A be a non empty compact subset of $\mathbb{R}$. The functor $\operatorname{divs} A$ is defined by:

(Def. 3) $x_1 \in \operatorname{divs} A$ iff x_1 is a DivisionPoint of A.

Let A be a non empty compact subset of $\mathbb{R}$. One can verify that $\operatorname{divs} A$ is non empty.

Let A be a non empty compact subset of $\mathbb{R}$. A non empty set is called a Division of A if:

(Def. 4) $x_1 \in \mathrm{it}$ iff x_1 is a DivisionPoint of A.

Let A be a non empty compact subset of $\mathbb{R}$. Observe that there exists a Division of A which is non empty.

Let A be a non empty compact subset of $\mathbb{R}$ and let S be a non empty Division of A. We see that the element of S is a DivisionPoint of A.

In the sequel S denotes a non empty Division of A and D denotes an element of S.

The following propositions are true:

(8)[1] If $i \in \operatorname{dom} D$, then $D(i) \in A$.

(9) If $i \in \operatorname{dom} D$ and $i \neq 1$, then $i - 1 \in \operatorname{dom} D$ and $D(i-1) \in A$ and $i - 1 \in \mathbb{N}$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, let D be an element of S, and let i be a natural number. Let us assume that $i \in \operatorname{dom} D$. The functor $\operatorname{divset}(D, i)$ yielding a closed-interval subset of $\mathbb{R}$ is defined as follows:

(Def. 5)(i) $\inf \operatorname{divset}(D, i) = \inf A$ and $\sup \operatorname{divset}(D, i) = D(i)$ if $i = 1$,

(ii) $\inf \operatorname{divset}(D, i) = D(i-1)$ and $\sup \operatorname{divset}(D, i) = D(i)$, otherwise.

Next we state the proposition

(10) If $i \in \operatorname{dom} D$, then $\operatorname{divset}(D, i) \subseteq A$.

Let A be a subset of $\mathbb{R}$. The functor $\operatorname{vol}(A)$ yielding a real number is defined as follows:

(Def. 6) $\operatorname{vol}(A) = \sup A - \inf A$.

We now state the proposition

(11) For every bounded non empty subset A of $\mathbb{R}$ holds $0 \leq \operatorname{vol}(A)$.

3. DEFINITIONS OF INTEGRABILITY AND RELATED TOPICS

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. The functor $\mathrm{upper\_volume}(f, D)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 7) $\operatorname{len} \mathrm{upper\_volume}(f, D) = \operatorname{len} D$ and for every i such that $i \in \operatorname{Seg} \operatorname{len} D$ holds $(\mathrm{upper\_volume}(f, D))(i) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

[1] The proposition (7) has been removed.

The functor $\mathrm{lower\_volume}(f,D)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 8) $\mathrm{len}\,\mathrm{lower\_volume}(f,D) = \mathrm{len}\,D$ and for every i such that $i \in \mathrm{Seg}\,\mathrm{len}\,D$ holds $(\mathrm{lower\_volume}(f,D))(i) = \inf \mathrm{rng}(f \restriction \mathrm{divset}(D,i)) \cdot \mathrm{vol}(\mathrm{divset}(D,i))$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. The functor $\mathrm{upper\_sum}(f,D)$ yielding a real number is defined by:

(Def. 9) $\mathrm{upper\_sum}(f,D) = \sum \mathrm{upper\_volume}(f,D)$.

The functor $\mathrm{lower\_sum}(f,D)$ yielding a real number is defined as follows:

(Def. 10) $\mathrm{lower\_sum}(f,D) = \sum \mathrm{lower\_volume}(f,D)$.

Let A be a closed-interval subset of $\mathbb{R}$. Then $\mathrm{divs}\,A$ is a Division of A.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor $\mathrm{upper\_sum\_set}\,f$ yielding a partial function from $\mathrm{divs}\,A$ to $\mathbb{R}$ is defined by:

(Def. 11) $\mathrm{dom}\,\mathrm{upper\_sum\_set}\,f = \mathrm{divs}\,A$ and for every element D of $\mathrm{divs}\,A$ such that $D \in \mathrm{dom}\,\mathrm{upper\_sum\_set}\,f$ holds $(\mathrm{upper\_sum\_set}\,f)(D) = \mathrm{upper\_sum}(f,D)$.

The functor $\mathrm{lower\_sum\_set}\,f$ yields a partial function from $\mathrm{divs}\,A$ to $\mathbb{R}$ and is defined by:

(Def. 12) $\mathrm{dom}\,\mathrm{lower\_sum\_set}\,f = \mathrm{divs}\,A$ and for every element D of $\mathrm{divs}\,A$ such that $D \in \mathrm{dom}\,\mathrm{lower\_sum\_set}\,f$ holds $(\mathrm{lower\_sum\_set}\,f)(D) = \mathrm{lower\_sum}(f,D)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. We say that f is upper integrable on A if and only if:

(Def. 13) $\mathrm{rng}\,\mathrm{upper\_sum\_set}\,f$ is lower bounded.

We say that f is lower integrable on A if and only if:

(Def. 14) $\mathrm{rng}\,\mathrm{lower\_sum\_set}\,f$ is upper bounded.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor $\mathrm{upper\_integral}\,f$ yielding a real number is defined by:

(Def. 15) $\mathrm{upper\_integral}\,f = \inf \mathrm{rng}\,\mathrm{upper\_sum\_set}\,f$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor $\mathrm{lower\_integral}\,f$ yielding a real number is defined by:

(Def. 16) $\mathrm{lower\_integral}\,f = \sup \mathrm{rng}\,\mathrm{lower\_sum\_set}\,f$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. We say that f is integrable on A if and only if:

(Def. 17) f is upper integrable on A and f is lower integrable on A and $\mathrm{upper\_integral}\,f = \mathrm{lower\_integral}\,f$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor $\mathrm{integral}\,f$ yielding a real number is defined by:

(Def. 18) $\mathrm{integral}\,f = \mathrm{upper\_integral}\,f$.

4. Real Function's Properties

One can prove the following propositions:

(12) For all partial functions f, g from X to $\mathbb{R}$ holds $\mathrm{rng}(f+g) \subseteq \mathrm{rng}\, f + \mathrm{rng}\, g$.

(13) For every partial function f from X to $\mathbb{R}$ such that f is lower bounded on X holds $\mathrm{rng}\, f$ is lower bounded.

(14) For every partial function f from X to $\mathbb{R}$ such that $\mathrm{rng}\, f$ is lower bounded holds f is lower bounded on X.

(15) For every partial function f from X to $\mathbb{R}$ such that f is upper bounded on X holds $\mathrm{rng}\, f$ is upper bounded.

(16) For every partial function f from X to $\mathbb{R}$ such that $\mathrm{rng}\, f$ is upper bounded holds f is upper bounded on X.

(17) For every partial function f from X to $\mathbb{R}$ such that f is bounded on X holds $\mathrm{rng}\, f$ is bounded.

5. Characteristic Function's Properties

The following propositions are true:

(18) For every non empty set A holds $\chi_{A,A}$ is a constant on A.

(19) For every non empty subset A of X holds $\mathrm{rng}(\chi_{A,A}) = \{1\}$.

(20) For every non empty subset A of X and for every set B such that B meets $\mathrm{dom}(\chi_{A,A})$ holds $\mathrm{rng}(\chi_{A,A} \restriction B) = \{1\}$.

(21) If $i \in \mathrm{Seg}\,\mathrm{len}\, D$, then $\mathrm{vol}(\mathrm{divset}(D,i)) = (\mathrm{lower\_volume}(\chi_{A,A}, D))(i)$.

(22) If $i \in \mathrm{Seg}\,\mathrm{len}\, D$, then $\mathrm{vol}(\mathrm{divset}(D,i)) = (\mathrm{upper\_volume}(\chi_{A,A}, D))(i)$.

(23) If $\mathrm{len}\, F = \mathrm{len}\, G$ and $\mathrm{len}\, F = \mathrm{len}\, H$ and for every k such that $k \in \mathrm{dom}\, F$ holds $H(k) = F_k + G_k$, then $\sum H = \sum F + \sum G$.

(24) If $\mathrm{len}\, F = \mathrm{len}\, G$ and $\mathrm{len}\, F = \mathrm{len}\, H$ and for every k such that $k \in \mathrm{dom}\, F$ holds $H(k) = F_k - G_k$, then $\sum H = \sum F - \sum G$.

(25) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $\sum \mathrm{lower\_volume}(\chi_{A,A}, D) = \mathrm{vol}(A)$.

(26) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $\sum \mathrm{upper\_volume}(\chi_{A,A}, D) = \mathrm{vol}(A)$.

6. Some Properties of Darboux Sum

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. Then $\mathrm{upper\_volume}(f,D)$ is a non empty finite sequence of elements of $\mathbb{R}$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. Then $\mathrm{lower\_volume}(f,D)$ is a non empty finite sequence of elements of $\mathbb{R}$.

One can prove the following propositions:

(27) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A, then $\inf \mathrm{rng}\, f \cdot \mathrm{vol}(A) \leq \mathrm{lower\_sum}(f,D)$.

(28) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and i be a natural number. If f is upper bounded on A and $i \in \operatorname{Seg} \operatorname{len} D$, then $\sup \operatorname{rng} f \cdot \operatorname{vol}(\operatorname{divset}(D,i)) \geq \sup \operatorname{rng}(f \upharpoonright \operatorname{divset}(D,i)) \cdot \operatorname{vol}(\operatorname{divset}(D,i))$.

(29) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A, then $\text{upper\_sum}(f,D) \leq \sup \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(30) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is bounded on A, then $\text{lower\_sum}(f,D) \leq \text{upper\_sum}(f,D)$.

Let x be a non empty finite sequence of elements of $\mathbb{R}$. Then $\operatorname{rng} x$ is a finite non empty subset of $\mathbb{R}$.

Let A be a closed-interval subset of $\mathbb{R}$ and let D be an element of $\operatorname{divs} A$. The functor δ_D yields a real number and is defined as follows:

(Def. 19) $\delta_D = \max \operatorname{rng} \text{upper\_volume}(\chi_{A,A}, D)$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, and let D_1, D_2 be elements of S. The predicate $D_1 \leq D_2$ is defined as follows:

(Def. 20) $\operatorname{len} D_1 \leq \operatorname{len} D_2$ and $\operatorname{rng} D_1 \subseteq \operatorname{rng} D_2$.

We introduce $D_2 \geq D_1$ as a synonym of $D_1 \leq D_2$.

Next we state several propositions:

(31) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $\operatorname{len} D_1 = 1$, then $D_1 \leq D_2$.

(32) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is upper bounded on A and $\operatorname{len} D_1 = 1$, then $\text{upper\_sum}(f,D_1) \geq \text{upper\_sum}(f,D_2)$.

(33) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is lower bounded on A and $\operatorname{len} D_1 = 1$, then $\text{lower\_sum}(f,D_1) \leq \text{lower\_sum}(f,D_2)$.

(34) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose $i \in \operatorname{dom} D$. Then there exist closed-interval subsets A_1, A_2 of $\mathbb{R}$ such that $A_1 = [\inf A, D(i)]$ and $A_2 = [D(i), \sup A]$ and $A = A_1 \cup A_2$.

(35) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $i \in \operatorname{dom} D_1$, then if $D_1 \leq D_2$, then there exists j such that $j \in \operatorname{dom} D_2$ and $D_1(i) = D_2(j)$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, let D_1, D_2 be elements of S, and let i be a natural number. Let us assume that $D_1 \leq D_2$. The functor $\operatorname{indx}(D_2,D_1,i)$ yielding a natural number is defined as follows:

(Def. 21)(i) $\operatorname{indx}(D_2,D_1,i) \in \operatorname{dom} D_2$ and $D_1(i) = D_2(\operatorname{indx}(D_2,D_1,i))$ if $i \in \operatorname{dom} D_1$,

(ii) $\operatorname{indx}(D_2,D_1,i) = 0$, otherwise.

We now state four propositions:

(36) Let p be an increasing finite sequence of elements of $\mathbb{R}$ and n be a natural number. Suppose $n \leq \operatorname{len} p$. Then $p_{\restriction n}$ is an increasing finite sequence of elements of $\mathbb{R}$.

(37) Let p be an increasing finite sequence of elements of $\mathbb{R}$ and i, j be natural numbers. Suppose $j \in \operatorname{dom} p$ and $i \leq j$. Then $\operatorname{mid}(p,i,j)$ is an increasing finite sequence of elements of $\mathbb{R}$.

(38) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and i, j be natural numbers. Suppose $i \in \mathrm{dom} D$ and $j \in \mathrm{dom} D$ and $i \leq j$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $\inf B = (\mathrm{mid}(D,i,j))(1)$ and $\sup B = (\mathrm{mid}(D,i,j))(\mathrm{len}\,\mathrm{mid}(D,i,j))$ and $\mathrm{len}\,\mathrm{mid}(D,i,j) = (j-i)+1$ and $\mathrm{mid}(D,i,j)$ is a DivisionPoint of B.

(39) Let A, B be closed-interval subsets of $\mathbb{R}$, S be a non empty Division of A, S_1 be a non empty Division of B, D be an element of S, and i, j be natural numbers. Suppose $i \in \mathrm{dom} D$ and $j \in \mathrm{dom} D$ and $i \leq j$ and $D(i) \geq \inf B$ and $D(j) = \sup B$. Then $\mathrm{mid}(D,i,j)$ is an element of S_1.

Let p be a finite sequence of elements of $\mathbb{R}$. The functor PartSums p yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 22) $\mathrm{len}\,\mathrm{PartSums}\,p = \mathrm{len}\,p$ and for every i such that $i \in \mathrm{Seg}\,\mathrm{len}\,p$ holds $(\mathrm{PartSums}\,p)(i) = \sum(p \restriction i)$.

We now state a number of propositions:

(40) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Suppose $D_1 \leq D_2$ and f is upper bounded on A. Let i be a non empty natural number. If $i \in \mathrm{dom} D_1$, then $\sum(\mathrm{upper\_volume}(f,D_1)\restriction i) \geq \sum(\mathrm{upper\_volume}(f,D_2)\restriction \mathrm{indx}(D_2,D_1,i))$.

(41) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Suppose $D_1 \leq D_2$ and f is lower bounded on A. Let i be a non empty natural number. If $i \in \mathrm{dom} D_1$, then $\sum(\mathrm{lower\_volume}(f,D_1)\restriction i) \leq \sum(\mathrm{lower\_volume}(f,D_2)\restriction \mathrm{indx}(D_2,D_1,i))$.

(42) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D_1, D_2 be elements of S, and i be a natural number. If $D_1 \leq D_2$ and $i \in \mathrm{dom} D_1$ and f is upper bounded on A, then $(\mathrm{PartSums}\,\mathrm{upper\_volume}(f,D_1))(i) \geq (\mathrm{PartSums}\,\mathrm{upper\_volume}(f,D_2))(\mathrm{indx}(D_2,D_1,i))$.

(43) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D_1, D_2 be elements of S, and i be a natural number. If $D_1 \leq D_2$ and $i \in \mathrm{dom} D_1$ and f is lower bounded on A, then $(\mathrm{PartSums}\,\mathrm{lower\_volume}(f,D_1))(i) \leq (\mathrm{PartSums}\,\mathrm{lower\_volume}(f,D_2))(\mathrm{indx}(D_2,D_1,i))$.

(44) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $(\mathrm{PartSums}\,\mathrm{upper\_volume}(f,D))(\mathrm{len}\,D) = \mathrm{upper\_sum}(f,D)$.

(45) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $(\mathrm{PartSums}\,\mathrm{lower\_volume}(f,D))(\mathrm{len}\,D) = \mathrm{lower\_sum}(f,D)$.

(46) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leq D_2$, then $\mathrm{indx}(D_2,D_1,\mathrm{len}\,D_1) = \mathrm{len}\,D_2$.

(47) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leq D_2$ and f is upper bounded on A, then $\mathrm{upper\_sum}(f,D_2) \leq \mathrm{upper\_sum}(f,D_1)$.

(48) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leq D_2$ and f is lower bounded on A, then $\mathrm{lower\_sum}(f,D_2) \geq \mathrm{lower\_sum}(f,D_1)$.

(49) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Then there exists an element D of S such that $D_1 \leq D$ and $D_2 \leq D$.

(50) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is bounded on A, then $\mathrm{lower\_sum}(f,D_1) \leq \mathrm{upper\_sum}(f,D_2)$.

7. Additivity of Integral

Next we state several propositions:

(51) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. If f is bounded on A, then $\mathrm{upper\_integral}\, f \geq \mathrm{lower\_integral}\, f$.

(52) For all subsets X, Y of $\mathbb{R}$ holds $-X + -Y = -(X+Y)$.

(53) For all subsets X, Y of $\mathbb{R}$ such that X is upper bounded and Y is upper bounded holds $X+Y$ is upper bounded.

(54) For all non empty subsets X, Y of $\mathbb{R}$ such that X is upper bounded and Y is upper bounded holds $\sup(X+Y) = \sup X + \sup Y$.

(55) Let A be a closed-interval subset of $\mathbb{R}$, f, g be functions from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, D$ and f is upper bounded on A and g is upper bounded on A. Then $(\mathrm{upper\_volume}(f+g,D))(i) \leq (\mathrm{upper\_volume}(f,D))(i) + (\mathrm{upper\_volume}(g,D))(i)$.

(56) Let A be a closed-interval subset of $\mathbb{R}$, f, g be functions from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, D$ and f is lower bounded on A and g is lower bounded on A. Then $(\mathrm{lower\_volume}(f,D))(i) + (\mathrm{lower\_volume}(g,D))(i) \leq (\mathrm{lower\_volume}(f+g,D))(i)$.

(57) Let A be a closed-interval subset of $\mathbb{R}$, f, g be functions from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose f is upper bounded on A and g is upper bounded on A. Then $\mathrm{upper\_sum}(f+g,D) \leq \mathrm{upper\_sum}(f,D) + \mathrm{upper\_sum}(g,D)$.

(58) Let A be a closed-interval subset of $\mathbb{R}$, f, g be functions from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose f is lower bounded on A and g is lower bounded on A. Then $\mathrm{lower\_sum}(f,D) + \mathrm{lower\_sum}(g,D) \leq \mathrm{lower\_sum}(f+g,D)$.

(59) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be functions from A into $\mathbb{R}$. Suppose f is bounded on A and g is bounded on A and f is integrable on A and g is integrable on A. Then $f+g$ is integrable on A and $\mathrm{integral}\, f+g = \mathrm{integral}\, f + \mathrm{integral}\, g$.

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