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Integers

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Summary. In the article the following concepts were introduced: the set of integers ($\mathbb{Z}$) and its elements (integers), congruences ($i_1 \equiv i_2 (\mathrm{mod}\, i_3)$), the ceiling and floor functors ($\lceil x \rceil$ and $\lfloor x \rfloor$), also the fraction part of a real number (frac), the integer division ($\div$) and remainder of integer division (mod). The following schemes were also included: the separation scheme (*SepInt*), the schemes of integer induction (*Int_Ind_Down*, *Int_Ind_Up*, *Int_Ind_Full*), the minimum (*Int_Min*) and maximum (*Int_Max*) schemes (the existence of minimum and maximum integers enjoying a given property).

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The articles [6], [3], [8], [1], [2], [7], [4], and [5] provide the notation and terminology for this paper.

We adopt the following rules: x denotes a set, k, n_1, n_2 denote natural numbers, and r denotes a real number.

Let us observe that every element of $\mathbb{C}$ is complex.

$\mathbb{Z}$ can be characterized by the condition:

(Def. 1) $x \in \mathbb{Z}$ iff there exists k such that $x = k$ or $x = -k$.

Let i be a number. We say that i is integer if and only if:

(Def. 2) i is an element of $\mathbb{Z}$.

One can check the following observations:

* there exists a real number which is integer,
* there exists a number which is integer, and
* every element of $\mathbb{Z}$ is integer.

An integer is an integer number.

One can prove the following proposition

(8)[1] r is an integer iff there exists k such that $r = k$ or $r = -k$.

Let us note that every natural number is integer and every number which is natural is also integer.

We now state the proposition

(11)[2] If $x \in \mathbb{Z}$, then $x \in \mathbb{R}$.

[1] The propositions (1)–(7) have been removed.
[2] The propositions (9) and (10) have been removed.

Let us observe that every number which is integer is also real.

We now state three propositions:

(12) x is an integer iff $x \in \mathbb{Z}$.

(14)[3] $\mathbb{N} \subseteq \mathbb{Z}$.

(15) $\mathbb{Z} \subseteq \mathbb{R}$.

In the sequel i_0, i_1, i_2, i_3, i_4, i_5 denote integers.

Let i_1, i_2 be integers. Observe that $i_1 + i_2$ is integer and $i_1 \cdot i_2$ is integer.

Let i_0 be an integer. Observe that $-i_0$ is integer.

Let i_1, i_2 be integers. One can check that $i_1 - i_2$ is integer.

Let n be a natural number. Observe that $-n$ is integer. Let i_1 be an integer. One can check the following observations:

* $i_1 + n$ is integer,
* $i_1 \cdot n$ is integer, and
* $i_1 - n$ is integer.

Let us consider n_1, n_2. Observe that $n_1 - n_2$ is integer.

One can prove the following propositions:

(16) If $0 \leq i_0$, then i_0 is a natural number.

(17) If r is an integer, then $r + 1$ is an integer and $r - 1$ is an integer.

(18) If $i_2 \leq i_1$, then $i_1 - i_2$ is a natural number.

(19) If $i_1 + k = i_2$, then $i_1 \leq i_2$.

(20) If $i_0 < i_1$, then $i_0 + 1 \leq i_1$.

(21) If $i_1 < 0$, then $i_1 \leq -1$.

(22) $i_1 \cdot i_2 = 1$ iff $i_1 = 1$ and $i_2 = 1$ or $i_1 = -1$ and $i_2 = -1$.

(23) $i_1 \cdot i_2 = -1$ iff $i_1 = -1$ and $i_2 = 1$ or $i_1 = 1$ and $i_2 = -1$.

(26)[4] $r - 1 < r$.

In this article we present several logical schemes. The scheme *SepInt* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a subset X of $\mathbb{Z}$ such that for every integer x holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

The scheme *Int Ind Up* deals with an integer $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every i_0 such that $\mathcal{A} \leq i_0$ holds $\mathcal{P}[i_0]$

provided the following conditions are satisfied:

- $\mathcal{P}[\mathcal{A}]$, and
- For every i_2 such that $\mathcal{A} \leq i_2$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 + 1]$.

The scheme *Int Ind Down* deals with an integer $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every i_0 such that $i_0 \leq \mathcal{A}$ holds $\mathcal{P}[i_0]$

provided the parameters meet the following requirements:

- $\mathcal{P}[\mathcal{A}]$, and
- For every i_2 such that $i_2 \leq \mathcal{A}$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 - 1]$.

The scheme *Int Ind Full* deals with an integer $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every i_0 holds $\mathcal{P}[i_0]$

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[3] The proposition (13) has been removed.

[4] The propositions (24) and (25) have been removed.

provided the following requirements are met:

- $\mathcal{P}[\mathcal{A}]$, and
- For every i_2 such that $\mathcal{P}[i_2]$ holds $\mathcal{P}[i_2 - 1]$ and $\mathcal{P}[i_2 + 1]$.

The scheme *Int Min* deals with an integer $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists i_0 such that $\mathcal{P}[i_0]$ and for every i_1 such that $\mathcal{P}[i_1]$ holds $i_0 \leq i_1$

provided the parameters meet the following conditions:

- For every i_1 such that $\mathcal{P}[i_1]$ holds $\mathcal{A} \leq i_1$, and
- There exists i_1 such that $\mathcal{P}[i_1]$.

The scheme *Int Max* deals with an integer $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists i_0 such that $\mathcal{P}[i_0]$ and for every i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq i_0$

provided the following requirements are met:

- For every i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq \mathcal{A}$, and
- There exists i_1 such that $\mathcal{P}[i_1]$.

Let us consider r. One can verify that $\operatorname{sgn} r$ is integer.

One can prove the following propositions:

(29)[5] $\operatorname{sgn} r = 1$ or $\operatorname{sgn} r = -1$ or $\operatorname{sgn} r = 0$.

(30) $|r| = r$ or $|r| = -r$.

Let us consider i_0. One can check that $|i_0|$ is integer.

Let i_1, i_2, i_3 be integers. The predicate $i_1 \equiv i_2 (\operatorname{mod} i_3)$ is defined by:

(Def. 3) There exists i_4 such that $i_3 \cdot i_4 = i_1 - i_2$.

The following propositions are true:

(32)[6] $i_1 \equiv i_1 (\operatorname{mod} i_2)$.

(33) $i_1 \equiv 0 (\operatorname{mod} i_1)$ and $0 \equiv i_1 (\operatorname{mod} i_1)$.

(34) $i_1 \equiv i_2 (\operatorname{mod} 1)$.

(35) If $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_2 \equiv i_1 (\operatorname{mod} i_3)$.

(36) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_2 \equiv i_3 (\operatorname{mod} i_5)$, then $i_1 \equiv i_3 (\operatorname{mod} i_5)$.

(37) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 + i_3 \equiv i_2 + i_4 (\operatorname{mod} i_5)$.

(38) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 - i_3 \equiv i_2 - i_4 (\operatorname{mod} i_5)$.

(39) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 \cdot i_3 \equiv i_2 \cdot i_4 (\operatorname{mod} i_5)$.

(40) $i_1 + i_2 \equiv i_3 (\operatorname{mod} i_5)$ iff $i_1 \equiv i_3 - i_2 (\operatorname{mod} i_5)$.

(41) If $i_4 \cdot i_5 = i_3$, then if $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_1 \equiv i_2 (\operatorname{mod} i_4)$.

(42) $i_1 \equiv i_2 (\operatorname{mod} i_5)$ iff $i_1 + i_5 \equiv i_2 (\operatorname{mod} i_5)$.

(43) $i_1 \equiv i_2 (\operatorname{mod} i_5)$ iff $i_1 - i_5 \equiv i_2 (\operatorname{mod} i_5)$.

(44) If $i_1 \leq r$ and $r - 1 < i_1$ and $i_2 \leq r$ and $r - 1 < i_2$, then $i_1 = i_2$.

(45) If $r \leq i_1$ and $i_1 < r + 1$ and $r \leq i_2$ and $i_2 < r + 1$, then $i_1 = i_2$.

Let r be a real number. The functor $\lfloor r \rfloor$ yielding an integer is defined as follows:

(Def. 4) $\lfloor r \rfloor \leq r$ and $r - 1 < \lfloor r \rfloor$.

Next we state several propositions:

[5] The propositions (27) and (28) have been removed.

[6] The proposition (31) has been removed.

(47)[7] $\lfloor r \rfloor = r$ iff r is integer.

(48) $\lfloor r \rfloor < r$ iff r is not integer.

(50)[8] $\lfloor r \rfloor - 1 < r$ and $\lfloor r \rfloor < r + 1$.

(51) $\lfloor r \rfloor + i_0 = \lfloor r + i_0 \rfloor$.

(52) $r < \lfloor r \rfloor + 1$.

Let r be a real number. The functor $\lceil r \rceil$ yields an integer and is defined as follows:

(Def. 5) $r \leq \lceil r \rceil$ and $\lceil r \rceil < r + 1$.

We now state a number of propositions:

(54)[9] $\lceil r \rceil = r$ iff r is integer.

(55) $r < \lceil r \rceil$ iff r is not integer.

(57)[10] $r - 1 < \lceil r \rceil$ and $r < \lceil r \rceil + 1$.

(58) $\lceil r \rceil + i_0 = \lceil r + i_0 \rceil$.

(59) $\lfloor r \rfloor = \lceil r \rceil$ iff r is integer.

(60) $\lfloor r \rfloor < \lceil r \rceil$ iff r is not integer.

(61) $\lfloor r \rfloor \leq \lceil r \rceil$.

(62) $\lfloor \lceil r \rceil \rfloor = \lceil r \rceil$.

(63) $\lfloor \lfloor r \rfloor \rfloor = \lfloor r \rfloor$.

(64) $\lceil \lceil r \rceil \rceil = \lceil r \rceil$.

(65) $\lceil \lfloor r \rfloor \rceil = \lfloor r \rfloor$.

(66) $\lfloor r \rfloor = \lceil r \rceil$ iff $\lfloor r \rfloor + 1 \neq \lceil r \rceil$.

Let r be a real number. The functor $\mathrm{frac}\, r$ is defined by:

(Def. 6) $\mathrm{frac}\, r = r - \lfloor r \rfloor$.

Let r be a real number. One can verify that $\mathrm{frac}\, r$ is real.
Let r be a real number. Then $\mathrm{frac}\, r$ is a real number.
We now state several propositions:

(68)[11] $r = \lfloor r \rfloor + \mathrm{frac}\, r$.

(69) $\mathrm{frac}\, r < 1$ and $0 \leq \mathrm{frac}\, r$.

(70) $\lfloor \mathrm{frac}\, r \rfloor = 0$.

(71) $\mathrm{frac}\, r = 0$ iff r is integer.

(72) $0 < \mathrm{frac}\, r$ iff r is not integer.

Let i_1, i_2 be integers. The functor $i_1 \div i_2$ yields an integer and is defined by:

(Def. 7) $i_1 \div i_2 = \lfloor \frac{i_1}{i_2} \rfloor$.

[7] The proposition (46) has been removed.
[8] The proposition (49) has been removed.
[9] The proposition (53) has been removed.
[10] The proposition (56) has been removed.
[11] The proposition (67) has been removed.

Let i_1, i_2 be integers. The functor $i_1 \bmod i_2$ yielding an integer is defined as follows:

(Def. 8) $$i_1 \bmod i_2 = \begin{cases} i_1 - (i_1 \div i_2) \cdot i_2, & \text{if } i_2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let i_1, i_2 be integers. The predicate $i_1 \mid i_2$ is defined as follows:

(Def. 9) There exists i_3 such that $i_2 = i_1 \cdot i_3$.

Let us note that the predicate $i_1 \mid i_2$ is reflexive.

Next we state four propositions:

(74)[12] For every real number r such that $r \neq 0$ holds $\lfloor \frac{r}{r} \rfloor = 1$.

(75) For every integer i holds $i \div 0 = 0$.

(76) For every integer i such that $i \neq 0$ holds $i \div i = 1$.

(77) For every integer i holds $i \bmod i = 0$.

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[12] The proposition (73) has been removed.