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# Axioms of Incidency

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**Summary.** This article is based on *"Foundations of Geometry"* by Karol Borsuk and Wanda Szmielew ([1]). The fourth axiom of incidency is weakened. In [1] it has the form *for any plane there exist three non-collinear points in the plane* and in the article *for any plane there exists one point in the plane*. The original axiom is proved. The article includes: theorems concerning collinearity of points and coplanarity of points and lines, basic theorems concerning lines and planes, fundamental existence theorems, theorems concerning intersection of lines and planes.

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The articles [5], [4], [2], [6], [3], and [7] provide the notation and terminology for this paper.

We consider projective incidence structures as systems

⟨ points, lines, an incidence ⟩,

where the points and the lines constitute non empty sets and the incidence is a relation between the points and the lines.

We introduce incidence structures which are extensions of projective incidence structure and are systems

⟨ points, lines, planes, an incidence, an incidence2, an incidence3 ⟩,

where the points, the lines, and the planes constitute non empty sets, the incidence is a relation between the points and the lines, the incidence2 is a relation between the points and the planes, and the incidence3 is a relation between the lines and the planes.

Let $S$ be a projective incidence structure. A point of $S$ is an element of the points of $S$. A line of $S$ is an element of the lines of $S$.

Let $S$ be an incidence structure. A plane of $S$ is an element of the planes of $S$.

For simplicity, we adopt the following convention: $S$ is an incidence structure, $A$, $B$, $C$, $D$ are points of $S$, $L$ is a line of $S$, $P$ is a plane of $S$, and $F$, $G$ are subsets of the points of $S$.

Let $S$ be a projective incidence structure, let $A$ be a point of $S$, and let $L$ be a line of $S$. We say that $A$ lies on $L$ if and only if:

(Def. 1) $\langle A, L\rangle \in$ the incidence of $S$.

Let us consider $S$, let $A$ be a point of $S$, and let $P$ be a plane of $S$. We say that $A$ lies on $P$ if and only if:

(Def. 2) $\langle A, P\rangle \in$ the incidence2 of $S$.

Let us consider $S$, let $L$ be a line of $S$, and let $P$ be a plane of $S$. We say that $L$ lies on $P$ if and only if:

(Def. 3) $\langle L, P\rangle \in$ the incidence3 of $S$.

Let $S$ be a projective incidence structure, let $F$ be a subset of the points of $S$, and let $L$ be a line of $S$. We say that $F$ lies on $L$ if and only if:

(Def. 4) For every point $A$ of $S$ such that $A \in F$ holds $A$ lies on $L$.

Let us consider $S$, let $F$ be a subset of the points of $S$, and let $P$ be a plane of $S$. We say that $F$ lies on $P$ if and only if:

(Def. 5) For every $A$ such that $A \in F$ holds $A$ lies on $P$.

Let $S$ be a projective incidence structure and let $F$ be a subset of the points of $S$. We say that $F$ is linear if and only if:

(Def. 6) There exists a line $L$ of $S$ such that $F$ lies on $L$.

We introduce $F$ is linear as a synonym of $F$ is linear.

Let $S$ be an incidence structure and let $F$ be a subset of the points of $S$. We say that $F$ is planar if and only if:

(Def. 7) There exists a plane $P$ of $S$ such that $F$ lies on $P$.

We introduce $F$ is planar as a synonym of $F$ is planar.

We now state a number of propositions:

(11)[1] $\{A,B\}$ lies on $L$ iff $A$ lies on $L$ and $B$ lies on $L$.

(12) $\{A,B,C\}$ lies on $L$ iff $A$ lies on $L$ and $B$ lies on $L$ and $C$ lies on $L$.

(13) $\{A,B\}$ lies on $P$ iff $A$ lies on $P$ and $B$ lies on $P$.

(14) $\{A,B,C\}$ lies on $P$ iff $A$ lies on $P$ and $B$ lies on $P$ and $C$ lies on $P$.

(15) $\{A,B,C,D\}$ lies on $P$ iff $A$ lies on $P$ and $B$ lies on $P$ and $C$ lies on $P$ and $D$ lies on $P$.

(16) If $G \subseteq F$ and $F$ lies on $L$, then $G$ lies on $L$.

(17) If $G \subseteq F$ and $F$ lies on $P$, then $G$ lies on $P$.

(18) $F$ lies on $L$ and $A$ lies on $L$ iff $F \cup \{A\}$ lies on $L$.

(19) $F$ lies on $P$ and $A$ lies on $P$ iff $F \cup \{A\}$ lies on $P$.

(20) $F \cup G$ lies on $L$ iff $F$ lies on $L$ and $G$ lies on $L$.

(21) $F \cup G$ lies on $P$ iff $F$ lies on $P$ and $G$ lies on $P$.

(22) If $G \subseteq F$ and $F$ is linear, then $G$ is linear.

(23) If $G \subseteq F$ and $F$ is planar, then $G$ is planar.

Let $I_1$ be an incidence structure. We say that $I_1$ is incidence space-like if and only if the conditions (Def. 8) are satisfied.

(Def. 8) For every line $L$ of $I_1$ there exist points $A$, $B$ of $I_1$ such that $A \neq B$ and $\{A,B\}$ lies on $L$ and for all points $A$, $B$ of $I_1$ there exists a line $L$ of $I_1$ such that $\{A,B\}$ lies on $L$ and for all points $A$, $B$ of $I_1$ and for all lines $K$, $L$ of $I_1$ such that $A \neq B$ and $\{A,B\}$ lies on $K$ and $\{A,B\}$ lies on $L$ holds $K = L$ and for every plane $P$ of $I_1$ there exists a point $A$ of $I_1$ such that $A$ lies on $P$ and for all points $A$, $B$, $C$ of $I_1$ there exists a plane $P$ of $I_1$ such that $\{A,B,C\}$ lies on $P$ and for all points $A$, $B$, $C$ of $I_1$ and for all planes $P$, $Q$ of $I_1$ such that $\{A,B,C\}$ is not linear and $\{A,B,C\}$ lies on $P$ and $\{A,B,C\}$ lies on $Q$ holds $P = Q$ and for every line $L$ of $I_1$ and for every plane $P$ of $I_1$ such that there exist points $A$, $B$ of $I_1$ such that $A \neq B$ and $\{A,B\}$ lies on $L$ and $\{A,B\}$ lies on $P$ holds $L$ lies on $P$ and for every point $A$ of $I_1$ and for all planes $P$, $Q$ of $I_1$ such that $A$ lies on $P$ and $A$ lies on $Q$ there exists a point $B$ of $I_1$ such that $A \neq B$ and $B$ lies on $P$ and $B$ lies on $Q$ and there exist points $A$, $B$, $C$, $D$ of $I_1$ such that $\{A,B,C,D\}$ is not planar and for every point $A$ of $I_1$ and for every line $L$ of $I_1$ and for every plane $P$ of $I_1$ such that $A$ lies on $L$ and $L$ lies on $P$ holds $A$ lies on $P$.

[1] The propositions (1)–(10) have been removed.

Let us observe that there exists an incidence structure which is strict and incidence space-like.

An incidence space is an incidence space-like incidence structure.

For simplicity, we adopt the following convention: $S$ denotes an incidence space, $A$, $B$, $C$, $D$ denote points of $S$, $K$, $L$, $L_1$, $L_2$ denote lines of $S$, $P$, $Q$ denote planes of $S$, and $F$ denotes a subset of the points of $S$.

One can prove the following propositions:

(35)[2] If $F$ lies on $L$ and $L$ lies on $P$, then $F$ lies on $P$.

(36) $\{A,A,B\}$ is linear.

(37) $\{A,A,B,C\}$ is planar.

(38) If $\{A,B,C\}$ is linear, then $\{A,B,C,D\}$ is planar.

(39) If $A \neq B$ and $\{A,B\}$ lies on $L$ and $C$ does not lie on $L$, then $\{A,B,C\}$ is not linear.

(40) If $\{A,B,C\}$ is not linear and $\{A,B,C\}$ lies on $P$ and $D$ does not lie on $P$, then $\{A,B,C,D\}$ is not planar.

(41) If it is not true that there exists $P$ such that $K$ lies on $P$ and $L$ lies on $P$, then $K \neq L$.

(42) Suppose that

(i) it is not true that there exists $P$ such that $L$ lies on $P$ and $L_1$ lies on $P$ and $L_2$ lies on $P$, and

(ii) there exists $A$ such that $A$ lies on $L$ and $A$ lies on $L_1$ and $A$ lies on $L_2$.

Then $L \neq L_1$.

(43) Suppose $L_1$ lies on $P$ and $L_2$ lies on $P$ and $L$ does not lie on $P$ and $L_1 \neq L_2$. Then it is not true that there exists $Q$ such that $L$ lies on $Q$ and $L_1$ lies on $Q$ and $L_2$ lies on $Q$.

(44) There exists $P$ such that $A$ lies on $P$ and $L$ lies on $P$.

(45) If there exists $A$ such that $A$ lies on $K$ and $A$ lies on $L$, then there exists $P$ such that $K$ lies on $P$ and $L$ lies on $P$.

(46) If $A \neq B$, then there exists $L$ such that for every $K$ holds $\{A,B\}$ lies on $K$ iff $K = L$.

(47) If $\{A,B,C\}$ is not linear, then there exists $P$ such that for every $Q$ holds $\{A,B,C\}$ lies on $Q$ iff $P = Q$.

(48) If $A$ does not lie on $L$, then there exists $P$ such that for every $Q$ holds $A$ lies on $Q$ and $L$ lies on $Q$ iff $P = Q$.

(49) Suppose $K \neq L$ and there exists $A$ such that $A$ lies on $K$ and $A$ lies on $L$. Then there exists $P$ such that for every $Q$ holds $K$ lies on $Q$ and $L$ lies on $Q$ iff $P = Q$.

Let us consider $S$ and let us consider $A$, $B$. Let us assume that $A \neq B$. The functor $\mathrm{Line}(A,B)$ yields a line of $S$ and is defined by:

(Def. 9) $\{A,B\}$ lies on $\mathrm{Line}(A,B)$.

Let us consider $S$ and let us consider $A$, $B$, $C$. Let us assume that $\{A,B,C\}$ is not linear. The functor $\mathrm{Plane}(A,B,C)$ yielding a plane of $S$ is defined as follows:

(Def. 10) $\{A,B,C\}$ lies on $\mathrm{Plane}(A,B,C)$.

Let us consider $S$ and let us consider $A$, $L$. Let us assume that $A$ does not lie on $L$. The functor $\mathrm{Plane}(A,L)$ yielding a plane of $S$ is defined as follows:

(Def. 11) $A$ lies on $\mathrm{Plane}(A,L)$ and $L$ lies on $\mathrm{Plane}(A,L)$.

[2] The propositions (24)–(34) have been removed.

Let us consider $S$ and let us consider $K$, $L$. Let us assume that $K \neq L$ and there exists $A$ such that $A$ lies on $K$ and $A$ lies on $L$. The functor $\mathrm{Plane}(K,L)$ yielding a plane of $S$ is defined by:

(Def. 12) $K$ lies on $\mathrm{Plane}(K,L)$ and $L$ lies on $\mathrm{Plane}(K,L)$.

The following propositions are true:

(57)[3] If $A \neq B$, then $\mathrm{Line}(A,B) = \mathrm{Line}(B,A)$.

(58) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(A,C,B)$.

(59) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(B,A,C)$.

(60) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(B,C,A)$.

(61) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(C,A,B)$.

(62) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(C,B,A)$.

(64)[4] If $K \neq L$ and there exists $A$ such that $A$ lies on $K$ and $A$ lies on $L$, then $\mathrm{Plane}(K,L) = \mathrm{Plane}(L,K)$.

(65) If $A \neq B$ and $C$ lies on $\mathrm{Line}(A,B)$, then $\{A,B,C\}$ is linear.

(66) If $A \neq B$ and $A \neq C$ and $\{A,B,C\}$ is linear, then $\mathrm{Line}(A,B) = \mathrm{Line}(A,C)$.

(67) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(C,\mathrm{Line}(A,B))$.

(68) If $\{A,B,C\}$ is not linear and $D$ lies on $\mathrm{Plane}(A,B,C)$, then $\{A,B,C,D\}$ is planar.

(69) If $C$ does not lie on $L$ and $\{A,B\}$ lies on $L$ and $A \neq B$, then $\mathrm{Plane}(C,L) = \mathrm{Plane}(A,B,C)$.

(70) If $\{A,B,C\}$ is not linear, then $\mathrm{Plane}(A,B,C) = \mathrm{Plane}(\mathrm{Line}(A,B),\mathrm{Line}(A,C))$.

(71) There exist $A$, $B$, $C$ such that $\{A,B,C\}$ lies on $P$ and $\{A,B,C\}$ is not linear.

(72) There exist $A$, $B$, $C$, $D$ such that $A$ lies on $P$ and $\{A,B,C,D\}$ is not planar.

(73) There exists $B$ such that $A \neq B$ and $B$ lies on $L$.

(74) If $A \neq B$, then there exists $C$ such that $C$ lies on $P$ and $\{A,B,C\}$ is not linear.

(75) If $\{A,B,C\}$ is not linear, then there exists $D$ such that $\{A,B,C,D\}$ is not planar.

(76) There exist $B$, $C$ such that $\{B,C\}$ lies on $P$ and $\{A,B,C\}$ is not linear.

(77) If $A \neq B$, then there exist $C$, $D$ such that $\{A,B,C,D\}$ is not planar.

(78) There exist $B$, $C$, $D$ such that $\{A,B,C,D\}$ is not planar.

(79) There exists $L$ such that $A$ does not lie on $L$ and $L$ lies on $P$.

(80) Suppose $A$ lies on $P$. Then there exist $L$, $L_1$, $L_2$ such that $L_1 \neq L_2$ and $L_1$ lies on $P$ and $L_2$ lies on $P$ and $L$ does not lie on $P$ and $A$ lies on $L$ and $A$ lies on $L_1$ and $A$ lies on $L_2$.

(81) There exist $L$, $L_1$, $L_2$ such that

(i) $A$ lies on $L$,

(ii) $A$ lies on $L_1$,

(iii) $A$ lies on $L_2$, and

(iv) it is not true that there exists $P$ such that $L$ lies on $P$ and $L_1$ lies on $P$ and $L_2$ lies on $P$.

(82) There exists $P$ such that $A$ lies on $P$ and $L$ does not lie on $P$.

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[3] The propositions (50)–(56) have been removed.

[4] The proposition (63) has been removed.

(83) There exists $A$ such that $A$ lies on $P$ and $A$ does not lie on $L$.

(84) It is not true that there exists $K$ and there exists $P$ such that $L$ lies on $P$ and $K$ lies on $P$.

(85) There exist $P$, $Q$ such that $P \neq Q$ and $L$ lies on $P$ and $L$ lies on $Q$.

(87)[5] If $L$ does not lie on $P$ and $\{A,B\}$ lies on $L$ and $\{A,B\}$ lies on $P$, then $A = B$.

(88) Suppose $P \neq Q$. Then

(i) it is not true that there exists $A$ such that $A$ lies on $P$ and $A$ lies on $Q$, or

(ii) there exists $L$ such that for every $B$ holds $B$ lies on $P$ and $B$ lies on $Q$ iff $B$ lies on $L$.

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[5] The proposition (86) has been removed.