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Algebra of Normal Forms Is a Heyting Algebra[1]

Andrzej Trybulec
Warsaw University
Białystok

Summary. We prove that the lattice of normal forms over an arbitrary set, introduced in [12], is an implicative lattice. The relative pseudo-complement $\alpha \Rightarrow \beta$ is defined as $\bigsqcup_{\alpha_1 \cup \alpha_2 = \alpha} -\alpha_1 \sqcap \alpha_2 \rightarrowtail \beta$, where $-\alpha$ is the pseudo-complement of α and $\alpha \rightarrowtail \beta$ is a rather strong implication introduced in this paper.

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The articles [10], [6], [16], [17], [4], [5], [2], [11], [3], [7], [15], [8], [18], [13], [14], [9], [12], and [1] provide the notation and terminology for this paper.

The following proposition is true

(1) Let A, B, C be non empty sets and f be a function from A into B. Suppose that for every element x of A holds $f(x) \in C$. Then f is a function from A into C.

In the sequel A is a non empty set and a is an element of A.

Let us consider A and let B, C be elements of $\mathrm{Fin}\,A$. Let us observe that $B \subseteq C$ if and only if:

(Def. 1) For every a such that $a \in B$ holds $a \in C$.

Let A be a non empty set and let B be a non empty subset of A. Then $\overset{B}{\hookrightarrow}$ is a function from B into A.

In the sequel A denotes a set.

Let us consider A. Let us assume that A is non empty. The functor $[A]$ yields a non empty set and is defined as follows:

(Def. 2) $[A] = A$.

We adopt the following rules: B, C are elements of $\mathrm{Fin}\,\mathrm{DP}(A)$, a, b, c, s, t_1, t_2 are elements of $\mathrm{DP}(A)$, and u, v, w are elements of the lattice of normal forms over A.

Next we state the proposition

(3)[1] If $B = \emptyset$, then $\mu B = \emptyset$.

Let us consider A. Observe that there exists an element of the normal forms over A which is non empty.

Let us consider A, a. Then $\{a\}$ is an element of the normal forms over A.

Let us consider A and let u be an element of the lattice of normal forms over A. The functor ${}^{@}u$ yielding an element of the normal forms over A is defined by:

[1] Partially supported by RPBP.III-24.B1.

[1] The proposition (2) has been removed.

(Def. 3) $^{@}u = u$.

In the sequel K, L denote elements of the normal forms over A.
The following propositions are true:

(7)[2] $\mu(K \frown K) = K$.

(8) For every set X such that $X \subseteq K$ holds $X \in$ the normal forms over A.

(10)[3] For every set X such that $X \subseteq u$ holds X is an element of the lattice of normal forms over A.

Let us consider A. The functor $\{\Box\}_A$ yields a function from $\mathrm{DP}(A)$ into the carrier of the lattice of normal forms over A and is defined as follows:

(Def. 4) $\{\Box\}_A(a) = \{a\}$.

One can prove the following propositions:

(11) If $c \in \{\Box\}_A(a)$, then $c = a$.

(12) $a \in \{\Box\}_A(a)$.

(13) $\{\Box\}_A(a) = \mathrm{singleton}_{\mathrm{DP}(A)}(a)$.

(14) $\bigsqcup^{\mathrm{f}}_{K}(\{\Box\}_A) = \mathrm{FinUnion}(K, \mathrm{singleton}_{\mathrm{DP}(A)})$.

(15) $u = \bigsqcup^{\mathrm{f}}_{^{@}u}(\{\Box\}_A)$.

In the sequel f is an element of $[:\mathrm{Fin}A, \mathrm{Fin}A:]^{\mathrm{DP}(A)}$ and g is an element of $[A]^{\mathrm{DP}(A)}$.
Let A be a set. The functor $\Box \setminus_A \Box$ yielding a binary operation on $[:\mathrm{Fin}A, \mathrm{Fin}A:]$ is defined by:

(Def. 5) For all elements a, b of $[:\mathrm{Fin}A, \mathrm{Fin}A:]$ holds $\Box \setminus_A \Box(a, b) = a \setminus b$.

Let us consider A, B. The functor $-B$ yields an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ and is defined as follows:

(Def. 6) $-B = \mathrm{DP}(A) \cap \{\langle\{g(t_1) : g(t_1) \in (t_1)_{\mathbf{2}} \ \wedge\ t_1 \in B\}, \{g(t_2) : g(t_2) \in (t_2)_{\mathbf{1}} \ \wedge\ t_2 \in B\}\rangle : s \in B \ \Rightarrow\ g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}\}$.

Let us consider C. The functor $B \rightarrowtail C$ yielding an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ is defined as follows:

(Def. 7) $B \rightarrowtail C = \mathrm{DP}(A) \cap \{\mathrm{FinUnion}(B, \Box \setminus_A \Box^{\circ}(f, \overset{\mathrm{DP}(A)}{\hookrightarrow})) : f^{\circ}B \subseteq C\}$.

Next we state a number of propositions:

(16) Suppose $c \in -B$. Then there exists g such that for every s such that $s \in B$ holds $g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}$ and $c = \langle\{g(t_1) : g(t_1) \in (t_1)_{\mathbf{2}} \ \wedge\ t_1 \in B\}, \{g(t_2) : g(t_2) \in (t_2)_{\mathbf{1}} \ \wedge\ t_2 \in B\}\rangle$.

(17) $\langle\emptyset, \emptyset\rangle$ is an element of $\mathrm{DP}(A)$.

(18) For every K such that $K = \emptyset$ holds $-K = \{\langle\emptyset, \emptyset\rangle\}$.

(19) For all K, L such that $K = \emptyset$ and $L = \emptyset$ holds $K \rightarrowtail L = \{\langle\emptyset, \emptyset\rangle\}$.

(20) For every element a of $\mathrm{DP}(\emptyset)$ holds $a = \langle\emptyset, \emptyset\rangle$.

(21) $\mathrm{DP}(\emptyset) = \{\langle\emptyset, \emptyset\rangle\}$.

(22) $\{\langle\emptyset, \emptyset\rangle\}$ is an element of the normal forms over A.

(23) If $c \in B \rightarrowtail C$, then there exists f such that $f^{\circ}B \subseteq C$ and $c = \mathrm{FinUnion}(B, \Box \setminus_A \Box^{\circ}(f, \overset{\mathrm{DP}(A)}{\hookrightarrow}))$.

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[2] The propositions (4)–(6) have been removed.
[3] The proposition (9) has been removed.

(24) If $K \frown \{a\} = \emptyset$, then there exists b such that $b \in -K$ and $b \subseteq a$.

(25) Suppose for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and for every c such that $c \in u$ there exists b such that $b \in v$ and $b \subseteq c \cup a$. Then there exists b such that $b \in ({}^{@}u) \rightarrowtail {}^{@}v$ and $b \subseteq a$.

(26) $K \frown -K = \emptyset$.

Let us consider A. The functor $\square^{c}{}_{A}$ yields a unary operation on the carrier of the lattice of normal forms over A and is defined as follows:

(Def. 8) $\square^{c}{}_{A}(u) = \mu(-{}^{@}u)$.

The functor $\square \rightarrowtail_{A} \square$ yielding a binary operation on the carrier of the lattice of normal forms over A is defined by:

(Def. 9) $(\square \rightarrowtail_{A} \square)(u, v) = \mu(({}^{@}u) \rightarrowtail {}^{@}v)$.

Let us consider u. The functor 2^{u} yielding an element of Fin (the carrier of the lattice of normal forms over A) is defined by:

(Def. 10) $2^{u} = 2^{u}$.

The functor $\square \setminus_{u} \square$ yields a unary operation on the carrier of the lattice of normal forms over A and is defined as follows:

(Def. 11) $(\square \setminus_{u} \square)(v) = u \setminus v$.

One can prove the following propositions:

(27) $(\square \setminus_{u} \square)(v) \sqsubseteq u$.

(28) $u \sqcap \square^{c}{}_{A}(u) = \bot_{\text{the lattice of normal forms over } A}$.

(29) $u \sqcap (\square \rightarrowtail_{A} \square)(u, v) \sqsubseteq v$.

(30) If $({}^{@}u) \frown \{a\} = \emptyset$, then $\{\square\}_{A}(a) \sqsubseteq \square^{c}{}_{A}(u)$.

(31) If for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and $u \sqcap \{\square\}_{A}(a) \sqsubseteq w$, then $\{\square\}_{A}(a) \sqsubseteq (\square \rightarrowtail_{A} \square)(u, w)$.

Let us consider A. Observe that the lattice of normal forms over A is implicative.

Next we state two propositions:

(33)[4] $u \Rightarrow v = \bigsqcup^{\mathrm{f}}_{2^{u}}((\text{the meet operation of the lattice of normal forms over } A)^{\circ}(\square^{c}{}_{A}, (\square \rightarrowtail_{A} \square)^{\circ}(\square \setminus_{u} \square, v)))$.

(34) $\top_{\text{the lattice of normal forms over } A} = \{\langle \emptyset, \emptyset \rangle\}$.

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