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# Hermitan Functionals. Canonical Construction of Scalar Product in Quotient Vector Space[1]

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**Summary.** In the article we present antilinear functionals, sesquilinear and hermitan forms. We prove Schwarz and Minkowski inequalities, and Parallelogram Law for non negative hermitan form. The proof of Schwarz inequality is based on [16]. The incorrect proof of this fact can be found in [13]. The construction of scalar product in quotient vector space from non negative hermitan functions is the main result of the article.

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The articles [18], [5], [23], [1], [19], [7], [8], [17], [3], [2], [21], [12], [24], [4], [20], [6], [9], [22], [14], [15], [11], and [10] provide the notation and terminology for this paper.

## 1. Auxiliary Facts about Complex Numbers

One can prove the following propositions:

(1) For every element $a$ of $\mathbb{C}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(2) For every element $a$ of $\mathbb{C}$ such that $a \neq 0_{\mathbb{C}}$ holds $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = 0$.

(3) For every element $a$ of $\mathbb{C}$ there exists an element $b$ of $\mathbb{C}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(4) For every element $a$ of $\mathbb{C}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0i$.

(5) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(6) $\overline{i_{\mathbb{C}_{\mathrm{F}}}} = (i)^{-1}$.

(7) $i_{\mathbb{C}_{\mathrm{F}}} \cdot \overline{i_{\mathbb{C}_{\mathrm{F}}}} = \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(8) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ such that $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i_{\mathbb{C}_{\mathrm{F}}}| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = 0$.

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(9) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ there exists an element $b$ of $\mathbb{C}_{\mathrm{F}}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(10) For all elements $a$, $b$ of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a - b) = \Re(a) - \Re(b)$ and $\Im(a - b) = \Im(a) - \Im(b)$.

(11) For all elements $a$, $b$ of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $\Re(a \cdot b) = \Re(a) \cdot \Re(b)$ and $\Im(a \cdot b) = \Re(a) \cdot \Im(b)$.

(12) For all elements $a$, $b$ of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ and $\Im(b) = 0$ holds $\Im(a \cdot b) = 0$.

(13) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) = \Re(\overline{a})$.

(14) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $a = \overline{a}$.

(15) For all real numbers $r$, $s$ holds $(r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot (s + 0i_{\mathbb{C}_{\mathrm{F}}}) = r \cdot s + 0i_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0i_{\mathbb{C}_{\mathrm{F}}}$.

(17) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ such that $0 \leq \Re(a)$ and $\Im(a) = 0$ holds $|a| = \Re(a)$.

(18) For every element $a$ of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) + \Re(\overline{a}) = 2 \cdot \Re(a)$.

## 2. Antilinear Functionals in Complex Vector Spaces

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a functional in $V$. We say that $f$ is complex-homogeneous if and only if:

(Def. 1) For every vector $v$ of $V$ and for every scalar $a$ of $V$ holds $f(a \cdot v) = \overline{a} \cdot f(v)$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Note that 0Functional$V$ is complex-homogeneous.

Let $V$ be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every functional in $V$ which is complex-homogeneous is also 0-preserving.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that there exists a functional in $V$ which is additive, complex-homogeneous, and 0-preserving.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. An antilinear functional of $V$ is an additive complex-homogeneous functional in $V$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$, $g$ be complex-homogeneous functionals in $V$. Note that $f + g$ is complex-homogeneous.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a complex-homogeneous functional in $V$. Observe that $-f$ is complex-homogeneous.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let $a$ be a scalar of $V$, and let $f$ be a complex-homogeneous functional in $V$. Note that $a \cdot f$ is complex-homogeneous.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$, $g$ be complex-homogeneous functionals in $V$. Observe that $f - g$ is complex-homogeneous.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a functional in $V$. The functor $\overline{f}$ yielding a functional in $V$ is defined as follows:

(Def. 2) For every vector $v$ of $V$ holds $\overline{f}(v) = \overline{f(v)}$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be an additive functional in $V$. One can check that $\overline{f}$ is additive.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a homogeneous functional in $V$. Observe that $\overline{f}$ is complex-homogeneous.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a complex-homogeneous functional in $V$. One can verify that $\overline{f}$ is homogeneous.

Let $V$ be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a non constant functional in $V$. Note that $\overline{f}$ is non constant.

Let $V$ be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$. Note that there exists a functional in $V$ which is additive, complex-homogeneous, non constant, and non trivial.

Next we state a number of propositions:

(19) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for every functional $f$ in $V$ holds $\overline{\overline{f}} = f$.

(20) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ holds $\overline{0{\rm Functional}V} = 0{\rm Functional}V$.

(21) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for all functionals $f$, $g$ in $V$ holds $\overline{f+g} = \overline{f} + \overline{g}$.

(22) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for every functional $f$ in $V$ holds $\overline{-f} = -\overline{f}$.

(23) Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, $f$ be a functional in $V$, and $a$ be a scalar of $V$. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(24) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for all functionals $f$, $g$ in $V$ holds $\overline{f-g} = \overline{f} - \overline{g}$.

(25) Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, $f$ be a functional in $V$, and $v$ be a vector of $V$. Then $f(v) = 0_{\mathbb{C}_{\rm F}}$ if and only if $\overline{f}(v) = 0_{\mathbb{C}_{\rm F}}$.

(26) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for every functional $f$ in $V$ holds $\ker f = \ker \overline{f}$.

(27) Let $V$ be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$ and $f$ be an antilinear functional of $V$. Then $\ker f$ is linearly closed.

(28) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $W$ be a subspace of $V$, and $f$ be an antilinear functional of $V$. If the carrier of $W \subseteq \ker \overline{f}$, then ${}^{f}/_{W}$ is complex-homogeneous.

Let $V$ be a vector space over $\mathbb{C}_{\rm F}$ and let $f$ be an antilinear functional of $V$. The functor QcFunctional $f$ yielding an antilinear functional of ${}^{V}/_{{\rm Ker}\,\overline{f}}$ is defined as follows:

(Def. 3) QcFunctional $f = {}^{f}/_{{\rm Ker}\,\overline{f}}$.

One can prove the following proposition

(29) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be an antilinear functional of $V$, $A$ be a vector of ${}^{V}/_{{\rm Ker}\,\overline{f}}$, and $v$ be a vector of $V$. If $A = v + {\rm Ker}\,\overline{f}$, then $({\rm QcFunctional}\,f)(A) = f(v)$.

Let $V$ be a non trivial vector space over $\mathbb{C}_{\rm F}$ and let $f$ be a non constant antilinear functional of $V$. Note that QcFunctional $f$ is non constant.

Let $V$ be a vector space over $\mathbb{C}_{\rm F}$ and let $f$ be an antilinear functional of $V$. Observe that QcFunctional $f$ is non degenerated.

## 3. Sesquilinear Forms in Complex Vector Spaces

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be a form of $V$, $W$. We say that $f$ is complex-homogeneous wrt. second argument if and only if:

(Def. 4) For every vector $v$ of $V$ holds $f(v, \cdot)$ is complex-homogeneous.

Next we state the proposition

(30) Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$, $v$ be a vector of $V$, $w$ be a vector of $W$, $a$ be an element of $\mathbb{C}_{\rm F}$, and $f$ be a form of $V$, $W$. Suppose $f$ is complex-homogeneous wrt. second argument. Then $f(\langle v, a \cdot w\rangle) = \overline{a} \cdot f(\langle v, w\rangle)$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let $f$ be a form of $V$, $V$. We say that $f$ is hermitan if and only if:

(Def. 5) For all vectors $v$, $u$ of $V$ holds $f(\langle v, u\rangle) = \overline{f(\langle u, v\rangle)}$.

We say that $f$ is diagonal real valued if and only if:

(Def. 6) For every vector $v$ of $V$ holds $\Im(f(\langle v, v\rangle)) = 0$.

We say that $f$ is diagonal plus-real valued if and only if:

(Def. 7) For every vector $v$ of $V$ holds $0 \leq \Re(f(\langle v, v\rangle))$.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$. Observe that $\mathrm{NulForm}(V,W)$ is complex-homogeneous wrt. second argument.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that $\mathrm{NulForm}(V,V)$ is hermitan and $\mathrm{NulForm}(V,V)$ is diagonal plus-real valued.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that every form of $V$, $V$ which is hermitan is also diagonal real valued.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Observe that there exists a form of $V$, $V$ which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$. Observe that there exists a form of $V$, $W$ which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$. A sesquilinear form of $V$, $W$ is an additive wrt. first argument homogeneous wrt. first argument additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. One can verify that every form of $V$, $V$ which is hermitan and additive wrt. second argument is also additive wrt. first argument.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that every form of $V$, $V$ which is hermitan and additive wrt. first argument is also additive wrt. second argument.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Observe that every form of $V$, $V$ which is hermitan and homogeneous wrt. first argument is also complex-homogeneous wrt. second argument.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that every form of $V$, $V$ which is hermitan and complex-homogeneous wrt. second argument is also homogeneous wrt. first argument.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$. A hermitan form of $V$ is a hermitan additive wrt. first argument homogeneous wrt. first argument form of $V$, $V$.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$, let $f$ be a functional in $V$, and let $g$ be a complex-homogeneous functional in $W$. Observe that $f \otimes g$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$, let $f$ be a complex-homogeneous wrt. second argument form of $V$, $W$, and let $v$ be a vector of $V$. Observe that $f(v,\cdot)$ is complex-homogeneous.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$, $g$ be complex-homogeneous wrt. second argument forms of $V$, $W$. One can check that $f + g$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$, let $f$ be a complex-homogeneous wrt. second argument form of $V$, $W$, and let $a$ be a scalar of $V$. One can verify that $a \cdot f$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be a complex-homogeneous wrt. second argument form of $V$, $W$. Observe that $-f$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$, $g$ be complex-homogeneous wrt. second argument forms of $V$, $W$. Observe that $f - g$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non trivial vector spaces over $\mathbb{C}_{\rm F}$. One can check that there exists a form of $V$, $W$ which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be a form of $V$, $W$. The functor $\overline{f}$ yields a form of $V$, $W$ and is defined as follows:

(Def. 8) For every vector $v$ of $V$ and for every vector $w$ of $W$ holds $\overline{f}(\langle v, w\rangle) = \overline{f(\langle v, w\rangle)}$.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be an additive wrt. second argument form of $V$, $W$. Observe that $\overline{f}$ is additive wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be an additive wrt. first argument form of $V$, $W$. Observe that $\overline{f}$ is additive wrt. first argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be a homogeneous wrt. second argument form of $V$, $W$. Note that $\overline{f}$ is complex-homogeneous wrt. second argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let $f$ be a complex-homogeneous wrt. second argument form of $V$, $W$. One can verify that $\overline{f}$ is homogeneous wrt. second argument.

Let $V$, $W$ be non trivial vector spaces over $\mathbb{C}_{\rm F}$ and let $f$ be a non constant form of $V$, $W$. Observe that $\overline{f}$ is non constant.

One can prove the following proposition

(31) Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, $f$ be a functional in $V$, and $v$ be a vector of $V$. Then $f \otimes \overline{f}(\langle v, v\rangle) = |f(v)|^{\mathbf{2}} + 0i_{\mathbb{C}_{\rm F}}$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let $f$ be a functional in $V$. Observe that $f \otimes \overline{f}$ is diagonal plus-real valued, hermitan, and diagonal real valued.

Let $V$ be a non trivial vector space over $\mathbb{C}_{\rm F}$. One can check that there exists a form of $V$, $V$ which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

One can prove the following propositions:

(32) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ and for every form $f$ of $V$, $W$ holds $\overline{\overline{f}} = f$.

(33) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ holds $\overline{\mathrm{NulForm}(V,W)} = \mathrm{NulForm}(V,W)$.

(34) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ and for all forms $f$, $g$ of $V$, $W$ holds $\overline{f+g} = \overline{f} + \overline{g}$.

(35) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ and for every form $f$ of $V$, $W$ holds $\overline{-f} = -\overline{f}$.

(36) Let $V$, $W$ be non empty vector space structures over $\mathbb{C}_{\rm F}$, $f$ be a form of $V$, $W$, and $a$ be an element of $\mathbb{C}_{\rm F}$. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(37) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ and for all forms $f$, $g$ of $V$, $W$ holds $\overline{f-g} = \overline{f} - \overline{g}$.

(38) Let $V$, $W$ be vector spaces over $\mathbb{C}_{\rm F}$, $v$ be a vector of $V$, $w$, $t$ be vectors of $W$, and $f$ be an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$. Then $f(\langle v, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle)$.

(39) Let $V$, $W$ be vector spaces over $\mathbb{C}_{\rm F}$, $v$, $u$ be vectors of $V$, $w$, $t$ be vectors of $W$, and $f$ be a sesquilinear form of $V$, $W$. Then $f(\langle v-u, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle) - (f(\langle u, w\rangle) - f(\langle u, t\rangle))$.

(40) Let $V$, $W$ be add-associative right zeroed right complementable vector space-like non empty vector space structures over $\mathbb{C}_{\rm F}$, $v$, $u$ be vectors of $V$, $w$, $t$ be vectors of $W$, $a$, $b$ be elements of $\mathbb{C}_{\rm F}$, and $f$ be a sesquilinear form of $V$, $W$. Then $f(\langle v+a\cdot u, w+b\cdot t\rangle) = f(\langle v, w\rangle) + \overline{b} \cdot f(\langle v, t\rangle) + (a \cdot f(\langle u, w\rangle) + a \cdot (\overline{b} \cdot f(\langle u, t\rangle)))$.

(41) Let $V$, $W$ be vector spaces over $\mathbb{C}_{\rm F}$, $v$, $u$ be vectors of $V$, $w$, $t$ be vectors of $W$, $a$, $b$ be elements of $\mathbb{C}_{\rm F}$, and $f$ be a sesquilinear form of $V$, $W$. Then $f(\langle v-a\cdot u, w-b\cdot t\rangle) = f(\langle v, w\rangle) - \overline{b}\cdot f(\langle v, t\rangle) - (a\cdot f(\langle u, w\rangle) - a\cdot(\overline{b}\cdot f(\langle u, t\rangle)))$.

(42) Let $V$ be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$, $f$ be a complex-homogeneous wrt. second argument form of $V$, $V$, and $v$ be a vector of $V$. Then $f(\langle v, 0_V\rangle) = 0_{\mathbb{C}_{\rm F}}$.

(43) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $v$, $w$ be vectors of $V$, and $f$ be a hermitan form of $V$. Then $f(\langle v, w\rangle) + f(\langle v, w\rangle) + f(\langle v, w\rangle) + f(\langle v, w\rangle) = ((f(\langle v+w, v+w\rangle) - f(\langle v-w, v-w\rangle)) + i_{\mathbb{C}_{\rm F}}\cdot f(\langle v+i_{\mathbb{C}_{\rm F}}\cdot w, v+i_{\mathbb{C}_{\rm F}}\cdot w\rangle)) - i_{\mathbb{C}_{\rm F}}\cdot f(\langle v-i_{\mathbb{C}_{\rm F}}\cdot w, v-i_{\mathbb{C}_{\rm F}}\cdot w\rangle)$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, let $f$ be a form of $V$, $V$, and let $v$ be a vector of $V$. The functor $||v||_f^2$ yields a real number and is defined by:

(Def. 9) $||v||_f^2 = \Re(f(\langle v, v\rangle))$.

We now state a number of propositions:

(44) Let $V$ be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued diagonal real valued form of $V$, $V$, and $v$ be a vector of $V$. Then $|f(\langle v, v\rangle)| = \Re(f(\langle v, v\rangle))$ and $||v||_f^2 = |f(\langle v, v\rangle)|$.

(45) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $v$, $w$ be vectors of $V$, $f$ be a sesquilinear form of $V$, $V$, $r$ be a real number, and $a$ be an element of $\mathbb{C}_{\rm F}$. Suppose $|a| = 1$ and $\Re(a\cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a\cdot f(\langle w, v\rangle)) = 0$. Then $f(\langle v-(r+0i_{\mathbb{C}_{\rm F}})\cdot a\cdot w, v-(r+0i_{\mathbb{C}_{\rm F}})\cdot a\cdot w\rangle) = (f(\langle v, v\rangle) - (r+0i_{\mathbb{C}_{\rm F}})\cdot(a\cdot f(\langle w, v\rangle)) - (r+0i_{\mathbb{C}_{\rm F}})\cdot(\overline{a}\cdot f(\langle v, w\rangle))) + (r^{\mathbf{2}}+0i_{\mathbb{C}_{\rm F}})\cdot f(\langle w, w\rangle)$.

(46) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $v$, $w$ be vectors of $V$, $f$ be a diagonal plus-real valued hermitan form of $V$, $r$ be a real number, and $a$ be an element of $\mathbb{C}_{\rm F}$. Suppose $|a| = 1$ and $\Re(a\cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a\cdot f(\langle w, v\rangle)) = 0$. Then $\Re(f(\langle v-(r+0i_{\mathbb{C}_{\rm F}})\cdot a\cdot w, v-(r+0i_{\mathbb{C}_{\rm F}})\cdot a\cdot w\rangle)) = (||v||_f^2 - 2\cdot|f(\langle w, v\rangle)|\cdot r) + ||w||_f^2\cdot r^{\mathbf{2}}$ and $0 \leq (||v||_f^2 - 2\cdot|f(\langle w, v\rangle)|\cdot r) + ||w||_f^2\cdot r^{\mathbf{2}}$.

(47) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $v$, $w$ be vectors of $V$, and $f$ be a diagonal plus-real valued hermitan form of $V$. If $||w||_f^2 = 0$, then $|f(\langle w, v\rangle)| = 0$.

(48) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $v$, $w$ be vectors of $V$, and $f$ be a diagonal plus-real valued hermitan form of $V$. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leq ||v||_f^2\cdot||w||_f^2$.

(49) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued hermitan form of $V$, and $v$, $w$ be vectors of $V$. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leq |f(\langle v, v\rangle)|\cdot|f(\langle w, w\rangle)|$.

(50) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued hermitan form of $V$, and $v$, $w$ be vectors of $V$. Then $||v+w||_f^2 \leq (\sqrt{||v||_f^2} + \sqrt{||w||_f^2})^{\mathbf{2}}$.

(51) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued hermitan form of $V$, and $v$, $w$ be vectors of $V$. Then $|f(\langle v+w, v+w\rangle)| \leq (\sqrt{|f(\langle v, v\rangle)|} + \sqrt{|f(\langle w, w\rangle)|})^{\mathbf{2}}$.

(52) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a hermitan form of $V$, and $v$, $w$ be elements of $V$. Then $||v+w||_f^2 + ||v-w||_f^2 = 2\cdot||v||_f^2 + 2\cdot||w||_f^2$.

(53) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued hermitan form of $V$, and $v$, $w$ be elements of $V$. Then $|f(\langle v+w, v+w\rangle)| + |f(\langle v-w, v-w\rangle)| = 2\cdot|f(\langle v, v\rangle)| + 2\cdot|f(\langle w, w\rangle)|$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let $f$ be a form of $V$, $V$. The functor $||\cdot||_f$ yields a RFunctional of $V$ and is defined as follows:

(Def. 10) For every element $v$ of $V$ holds $(||\cdot||_f)(v) = \sqrt{||v||_f^2}$.

Let $V$ be a vector space over $\mathbb{C}_{\rm F}$ and let $f$ be a diagonal plus-real valued hermitan form of $V$. Then $||\cdot||_f$ is a Semi-Norm of $V$.

## 4. Kernel of Hermitan Forms and Hermitan Forms in Quotient Vector Spaces

Let $V$ be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$ and let $f$ be a complex-homogeneous wrt. second argument form of $V$, $V$. Observe that diagker $f$ is non empty.

Next we state several propositions:

(54) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$ and $f$ be a diagonal plus-real valued hermitan form of $V$. Then diagker $f$ is linearly closed.

(55) For every vector space $V$ over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form $f$ of $V$ holds diagker $f$ = leftker $f$.

(56) For every vector space $V$ over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form $f$ of $V$ holds diagker $f$ = rightker $f$.

(57) For every non empty vector space structure $V$ over $\mathbb{C}_{\rm F}$ and for every form $f$ of $V$, $V$ holds diagker $f$ = diagker $\overline{f}$.

(58) For all non empty vector space structures $V$, $W$ over $\mathbb{C}_{\rm F}$ and for every form $f$ of $V$, $W$ holds leftker $f$ = leftker $\overline{f}$ and rightker $f$ = rightker $\overline{f}$.

(59) For every vector space $V$ over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form $f$ of $V$ holds LKer $f$ = RKer $\overline{f}$.

(60) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued diagonal real valued form of $V$, $V$, and $v$ be a vector of $V$. If $\Re(f(\langle v, v\rangle)) = 0$, then $f(\langle v, v\rangle) = 0_{\mathbb{C}_{\rm F}}$.

(61) Let $V$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be a diagonal plus-real valued hermitan form of $V$, and $v$ be a vector of $V$. Suppose $\Re(f(\langle v, v\rangle)) = 0$ and $f$ is non degenerated on left and non degenerated on right. Then $v = 0_V$.

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, let $W$ be a vector space over $\mathbb{C}_{\rm F}$, and let $f$ be an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$. The functor $\mathrm{RQForm}^*(f)$ yields an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ and is defined as follows:

(Def. 11) $\mathrm{RQForm}^*(f) = \overline{\mathrm{RQForm}(\overline{f})}$.

Next we state the proposition

(62) Let $V$ be a non empty vector space structure over $\mathbb{C}_{\rm F}$, $W$ be a vector space over $\mathbb{C}_{\rm F}$, $f$ be an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$, $v$ be a vector of $V$, and $w$ be a vector of $W$. Then $(\mathrm{RQForm}^*(f))(\langle v, w + \mathrm{RKer}\,\overline{f}\rangle) = f(\langle v, w\rangle)$.

Let $V$, $W$ be vector spaces over $\mathbb{C}_{\rm F}$ and let $f$ be a sesquilinear form of $V$, $W$. Observe that $\mathrm{LQForm}(f)$ is additive wrt. second argument and complex-homogeneous wrt. second argument and $\mathrm{RQForm}^*(f)$ is additive wrt. first argument and homogeneous wrt. first argument.

Let $V$, $W$ be vector spaces over $\mathbb{C}_{\rm F}$ and let $f$ be a sesquilinear form of $V$, $W$. The functor $\mathrm{QForm}^* f$ yielding a sesquilinear form of ${}^{V}/_{\mathrm{LKer}\,f}$, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ is defined by the condition (Def. 12).

(Def. 12) Let $A$ be a vector of ${}^{V}/_{\mathrm{LKer}\,f}$, $B$ be a vector of ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$, $v$ be a vector of $V$, and $w$ be a vector of $W$. If $A = v + \mathrm{LKer}\,f$ and $B = w + \mathrm{RKer}\,\overline{f}$, then $(\mathrm{QForm}^* f)(\langle A, B\rangle) = f(\langle v, w\rangle)$.

Let $V$, $W$ be non trivial vector spaces over $\mathbb{C}_{\rm F}$ and let $f$ be a non constant sesquilinear form of $V$, $W$. One can verify that $\mathrm{QForm}^* f$ is non constant.

Let $V$ be a right zeroed non empty vector space structure over $\mathbb{C}_{\rm F}$, let $W$ be a vector space over $\mathbb{C}_{\rm F}$, and let $f$ be an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$. Note that $\mathrm{RQForm}^*(f)$ is non degenerated on right.

One can prove the following propositions:

(63) Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, $W$ be a vector space over $\mathbb{C}_{\mathrm{F}}$, and $f$ be an additive wrt. second argument complex-homogeneous wrt. second argument form of $V$, $W$. Then $\mathrm{leftker} f = \mathrm{leftker}(\mathrm{RQForm}^*(f))$.

(64) For all vector spaces $V$, $W$ over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form $f$ of $V$, $W$ holds $\mathrm{RKer}\,\overline{f} = \mathrm{RKer}\,\overline{\mathrm{LQForm}(f)}$.

(65) For all vector spaces $V$, $W$ over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form $f$ of $V$, $W$ holds $\mathrm{LKer} f = \mathrm{LKer}(\mathrm{RQForm}^*(f))$.

(66) For all vector spaces $V$, $W$ over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form $f$ of $V$, $W$ holds $\mathrm{QForm}^* f = \mathrm{RQForm}^*(\mathrm{LQForm}(f))$ and $\mathrm{QForm}^* f = \mathrm{LQForm}(\mathrm{RQForm}^*(f))$.

(67) Let $V$, $W$ be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and $f$ be a sesquilinear form of $V$, $W$. Then $\mathrm{leftker}(\mathrm{QForm}^* f) = \mathrm{leftker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}^* f) = \mathrm{rightker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and $\mathrm{leftker}(\mathrm{QForm}^* f) = \mathrm{leftker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$ and $\mathrm{rightker}(\mathrm{QForm}^* f) = \mathrm{rightker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$.

Let $V$, $W$ be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a sesquilinear form of $V$, $W$. One can check that $\mathrm{RQForm}^*(\mathrm{LQForm}(f))$ is non degenerated on left and non degenerated on right and $\mathrm{LQForm}(\mathrm{RQForm}^*(f))$ is non degenerated on left and non degenerated on right.

Let $V$, $W$ be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a sesquilinear form of $V$, $W$. Note that $\mathrm{QForm}^* f$ is non degenerated on left and non degenerated on right.

## 5. Scalar Product in Quotient Vector Space Generated by Nonnegative Hermitan Form

Let $V$ be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a form of $V$, $V$. We say that $f$ is positive diagonal valued if and only if:

(Def. 13) For every vector $v$ of $V$ such that $v \neq 0_V$ holds $0 < \Re(f(\langle v, v \rangle))$.

Let $V$ be a right zeroed non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of $V$, $V$ which is positive diagonal valued and additive wrt. first argument is also diagonal plus-real valued.

Let $V$ be a right zeroed non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can verify that every form of $V$, $V$ which is positive diagonal valued and additive wrt. second argument is also diagonal plus-real valued.

Let $V$ be a vector space over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a diagonal plus-real valued hermitan form of $V$. The functor $\langle\cdot|\cdot\rangle_f$ yielding a diagonal plus-real valued hermitan form of ${}^V/_{\mathrm{LKer} f}$ is defined as follows:

(Def. 14) $\langle\cdot|\cdot\rangle_f = \mathrm{QForm}^* f$.

We now state three propositions:

(68) Let $V$ be a vector space over $\mathbb{C}_{\mathrm{F}}$, $f$ be a diagonal plus-real valued hermitan form of $V$, $A$, $B$ be vectors of ${}^V/_{\mathrm{LKer} f}$, and $v$, $w$ be vectors of $V$. If $A = v + \mathrm{LKer} f$ and $B = w + \mathrm{LKer} f$, then $(\langle\cdot|\cdot\rangle_f)(\langle A, B \rangle) = f(\langle v, w \rangle)$.

(69) For every vector space $V$ over $\mathbb{C}_{\mathrm{F}}$ and for every diagonal plus-real valued hermitan form $f$ of $V$ holds $\mathrm{leftker}(\langle\cdot|\cdot\rangle_f) = \mathrm{leftker}(\mathrm{QForm}^* f)$.

(70) For every vector space $V$ over $\mathbb{C}_{\mathrm{F}}$ and for every diagonal plus-real valued hermitan form $f$ of $V$ holds $\mathrm{rightker}(\langle\cdot|\cdot\rangle_f) = \mathrm{rightker}(\mathrm{QForm}^* f)$.

Let $V$ be a vector space over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a diagonal plus-real valued hermitan form of $V$. Observe that $\langle\cdot|\cdot\rangle_f$ is non degenerated on left, non degenerated on right, and positive diagonal valued.

Let $V$ be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$ and let $f$ be a diagonal plus-real valued non constant hermitan form of $V$. Note that $\langle\cdot|\cdot\rangle_f$ is non constant.

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