JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Solvable Groups

Katarzyna Zawadzka
Warsaw University
Białystok

Summary. The concept of solvable group is introduced. Some theorems concerning heirdom of solvability are proved.

MML Identifier: GRSOLV_1.

WWW: http://mizar.org/JFM/Vol6/grsolv_1.html

The articles [6], [12], [13], [2], [3], [7], [5], [4], [1], [9], [10], [8], and [11] provide the notation and terminology for this paper.

In this paper i denotes a natural number.

Let I_1 be a group. We say that I_1 is solvable if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exists a finite sequence F of elements of $\mathrm{SubGr}\, I_1$ such that

(i) $\mathrm{len}\, F > 0$,

(ii) $F(1) = \Omega_{(I_1)}$,

(iii) $F(\mathrm{len}\, F) = \{\mathbf{1}\}_{(I_1)}$, and

(iv) for every i such that $i \in \mathrm{dom}\, F$ and $i+1 \in \mathrm{dom}\, F$ and for all strict subgroups G_1, G_2 of I_1 such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds ${}^{G_1}/_{N}$ is commutative.

One can verify that there exists a group which is solvable and strict.

The following propositions are true:

(1) Let G be a strict group and H, F_1, F_2 be strict subgroups of G. Suppose F_1 is a normal subgroup of F_2. Then $F_1 \cap H$ is a normal subgroup of $F_2 \cap H$.

(2) Let G be a strict group, F_2 be a strict subgroup of G, F_1 be a strict normal subgroup of F_2, and a, b be elements of F_2. Then $a \cdot F_1 \cdot (b \cdot F_1) = (a \cdot b) \cdot F_1$.

(3) Let G be a strict group, H, F_2 be strict subgroups of G, F_1 be a strict normal subgroup of F_2, and G_2 be a strict subgroup of G. Suppose $G_2 = H \cap F_2$. Let G_1 be a normal subgroup of G_2. Suppose $G_1 = H \cap F_1$. Then there exists a subgroup G_3 of ${}^{F_2}/_{F_1}$ such that ${}^{G_2}/_{G_1}$ and G_3 are isomorphic.

(4) Let G be a strict group, H, F_2 be strict subgroups of G, F_1 be a strict normal subgroup of F_2, and G_2 be a strict subgroup of G. Suppose $G_2 = F_2 \cap H$. Let G_1 be a normal subgroup of G_2. Suppose $G_1 = F_1 \cap H$. Then there exists a subgroup G_3 of ${}^{F_2}/_{F_1}$ such that ${}^{G_2}/_{G_1}$ and G_3 are isomorphic.

(5) For every solvable strict group G holds every strict subgroup of G is solvable.

(6) Let G be a strict group. Given a finite sequence F of elements of $\operatorname{SubGr} G$ such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_G$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and

(iv) for every i such that $i \in \operatorname{dom} F$ and $i+1 \in \operatorname{dom} F$ and for all strict subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds ${}^{G_1}/_{N}$ is a cyclic group.

Then G is solvable.

(7) Every strict commutative group is strict and solvable.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g \restriction A$ yielding a homomorphism from A to H is defined as follows:

(Def. 2) $g \restriction A = g \restriction$the carrier of A.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g^\circ A$ yields a strict subgroup of H and is defined as follows:

(Def. 3) $g^\circ A = \operatorname{Im}(g \restriction A)$.

The following propositions are true:

(8) Let G, H be strict groups, g be a homomorphism from G to H, and A be a subgroup of G. Then $\operatorname{rng}(g \restriction A) = g^\circ$(the carrier of A).

(9) Let G, H be strict groups, g be a homomorphism from G to H, and A be a strict subgroup of G. Then the carrier of $g^\circ A = g^\circ$(the carrier of A).

(10) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict subgroup of G. Then $\operatorname{Im}(h \restriction A)$ is a strict subgroup of $\operatorname{Im} h$.

(11) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict subgroup of G. Then $h^\circ A$ is a strict subgroup of $\operatorname{Im} h$.

(12) For all strict groups G, H and for every homomorphism h from G to H holds $h^\circ(\{\mathbf{1}\}_G) = \{\mathbf{1}\}_H$ and $h^\circ(\Omega_G) = \Omega_{\operatorname{Im} h}$.

(13) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be strict subgroups of G. If A is a subgroup of B, then $h^\circ A$ is a subgroup of $h^\circ B$.

(14) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, and a be an element of G. Then $h(a) \cdot h^\circ A = h^\circ(a \cdot A)$ and $h^\circ A \cdot h(a) = h^\circ(A \cdot a)$.

(15) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be subsets of G. Then $h^\circ A \cdot h^\circ B = h^\circ(A \cdot B)$.

(16) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be strict subgroups of G. Suppose A is a strict normal subgroup of B. Then $h^\circ A$ is a strict normal subgroup of $h^\circ B$.

(17) Let G, H be strict groups and h be a homomorphism from G to H. If G is a solvable group, then $\operatorname{Im} h$ is solvable.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[4] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/vectsp_1.html`.

[5] Dariusz Surowik. Cyclic groups and some of their properties — part I. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/gr_cy_1.html`.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[7] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[8] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_3.html`.

[9] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_1.html`.

[10] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_2.html`.

[11] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/group_6.html`.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received October 23, 1994

Published January 2, 2004