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# Oriented Chains

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**Summary.** In [8] we introduced a number of notions about vertex sequences associated with undirected chains of edges in graphs. In this article, we introduce analogous concepts for oriented chains and use them to prove properties of cutting and glueing of oriented chains, and the existence of a simple oriented chain in an oriented chain.

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The articles [10], [12], [13], [4], [3], [5], [11], [1], [6], [2], [7], [8], and [9] provide the notation and terminology for this paper.

## 1. Oriented Vertex Sequences

For simplicity, we use the following convention: $p$, $q$ are finite sequences, $e$, $X$ are sets, $m$, $n$ are natural numbers, $G$ is a graph, and $x$, $y$, $v$, $v_1$, $v_2$, $v_3$, $v_4$ are elements of the vertices of $G$.

Let us consider $G$, let us consider $x$, $y$, and let us consider $e$. We say that $e$ orientedly joins $x$, $y$ if and only if:

(Def. 1) (The source of $G)(e) = x$ and (the target of $G)(e) = y$.

Next we state the proposition

(1) If $e$ orientedly joins $v_1$, $v_2$, then $e$ joins $v_1$ with $v_2$.

Let us consider $G$ and let $x$, $y$ be elements of the vertices of $G$. We say that $x$, $y$ are orientedly incident if and only if:

(Def. 2) There exists a set $v$ such that $v \in$ the edges of $G$ and $v$ orientedly joins $x$, $y$.

The following proposition is true

(2) If $e$ orientedly joins $v_1$, $v_2$ and $e$ orientedly joins $v_3$, $v_4$, then $v_1 = v_3$ and $v_2 = v_4$.

We adopt the following rules: $v_5$, $v_6$, $v_7$ denote finite sequences of elements of the vertices of $G$ and $c$, $c_1$, $c_2$ denote oriented chains of $G$.

Let us consider $G$. One can check that there exists a chain of $G$ which is empty and oriented.

Let us consider $G$, $X$. The functor $G$-SVSet$X$ yielding a set is defined as follows:

(Def. 3) $G\text{-SVSet}X = \{v : \bigvee_{e:\text{element of the edges of } G} (e \in X \land v = (\text{the source of } G)(e))\}$.

Let us consider $G$, $X$. The functor $G$-TVSet$X$ yielding a set is defined as follows:

(Def. 4) $G\text{-TVSet}X = \{v : \bigvee_{e:\text{element of the edges of } G} (e \in X \land v = (\text{the target of } G)(e))\}$.

We now state the proposition

(4)[1] $G$-SVSet $\emptyset = \emptyset$ and $G$-TVSet $\emptyset = \emptyset$.

Let us consider $G$, $v_5$ and let $c$ be a finite sequence. We say that $v_5$ is oriented vertex seq of $c$ if and only if:

(Def. 5) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} c$ holds $c(n)$ orientedly joins $(v_5)_n$, $(v_5)_{n+1}$.

Next we state four propositions:

(5) If $v_5$ is oriented vertex seq of $c$, then $v_5$ is vertex sequence of $c$.

(6) If $v_5$ is oriented vertex seq of $c$, then $G$-SVSet $\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(7) If $v_5$ is oriented vertex seq of $c$, then $G$-TVSet $\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(8) If $c \neq \emptyset$ and $v_5$ is oriented vertex seq of $c$, then $\operatorname{rng} v_5 \subseteq (G\text{-SVSet} \operatorname{rng} c) \cup (G\text{-TVSet} \operatorname{rng} c)$.

## 2. Cutting and Glueing of Oriented Chains

One can prove the following three propositions:

(9) $\langle v \rangle$ is oriented vertex seq of $\emptyset$.

(10) There exists $v_5$ such that $v_5$ is oriented vertex seq of $c$.

(11) If $c \neq \emptyset$ and $v_6$ is oriented vertex seq of $c$ and $v_7$ is oriented vertex seq of $c$, then $v_6 = v_7$.

Let us consider $G$, $c$. Let us assume that $c \neq \emptyset$. The functor oriented-vertex-seq $c$ yields a finite sequence of elements of the vertices of $G$ and is defined by:

(Def. 6) oriented-vertex-seq $c$ is oriented vertex seq of $c$.

One can prove the following propositions:

(12) If $v_5$ is oriented vertex seq of $c$ and $c_1 = c \upharpoonright \operatorname{Seg} n$ and $v_6 = v_5 \upharpoonright \operatorname{Seg}(n+1)$, then $v_6$ is oriented vertex seq of $c_1$.

(13) If $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then $q$ is an oriented chain of $G$.

(14) Suppose $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and $v_5$ is oriented vertex seq of $c$ and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$. Then $v_6$ is oriented vertex seq of $c_1$.

(15) Suppose $v_6$ is oriented vertex seq of $c_1$ and $v_7$ is oriented vertex seq of $c_2$ and $v_6(\operatorname{len} v_6) = v_7(1)$. Then $c_1 \frown c_2$ is an oriented chain of $G$.

(16) Suppose $v_6$ is oriented vertex seq of $c_1$ and $v_7$ is oriented vertex seq of $c_2$ and $v_6(\operatorname{len} v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\!\frown v_7$. Then $v_5$ is oriented vertex seq of $c$.

[1] The proposition (3) has been removed.

## 3. ORIENTED SIMPLE CHAINS IN ORIENTED CHAINS

Let us consider $G$ and let $I_1$ be an oriented chain of $G$. We say that $I_1$ is Simple if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists $v_5$ such that $v_5$ is oriented vertex seq of $I_1$ and for all $n$, $m$ such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \operatorname{len} v_5$.

Let us consider $G$. One can verify that there exists an oriented chain of $G$ which is Simple.

Let us consider $G$. One can check that there exists a chain of $G$ which is oriented and simple.

We now state the proposition

(18)[2] For every oriented chain $q$ of $G$ holds $q \restriction \operatorname{Seg} n$ is an oriented chain of $G$.

In the sequel $s_1$ denotes an oriented simple chain of $G$.

One can prove the following propositions:

(19) $s_1 \restriction \operatorname{Seg} n$ is an oriented simple chain of $G$.

(20) For every oriented chain $s'_1$ of $G$ such that $s'_1 = s_1$ holds $s'_1$ is Simple.

(21) Every Simple oriented chain of $G$ is an oriented simple chain of $G$.

(22) Suppose $c$ is not Simple and $v_5$ is oriented vertex seq of $c$. Then there exists a FinSubsequence $f_1$ of $c$ and there exists a FinSubsequence $f_2$ of $v_5$ and there exist $c_1$, $v_6$ such that $\operatorname{len} c_1 < \operatorname{len} c$ and $v_6$ is oriented vertex seq of $c_1$ and $\operatorname{len} v_6 < \operatorname{len} v_5$ and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$ and $\operatorname{Seq} f_1 = c_1$ and $\operatorname{Seq} f_2 = v_6$.

(23) Suppose $v_5$ is oriented vertex seq of $c$. Then there exists a FinSubsequence $f_1$ of $c$ and there exists a FinSubsequence $f_2$ of $v_5$ and there exist $s_1$, $v_6$ such that $\operatorname{Seq} f_1 = s_1$ and $\operatorname{Seq} f_2 = v_6$ and $v_6$ is oriented vertex seq of $s_1$ and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$.

Let us consider $G$. Note that every oriented chain of $G$ which is empty is also one-to-one.

One can prove the following three propositions:

(24) If $p$ is an oriented path of $G$, then $p \restriction \operatorname{Seg} n$ is an oriented path of $G$.

(25) $s_1$ is an oriented path of $G$.

(26) Let $c_1$ be a finite sequence. Then

(i) $c_1$ is a Simple oriented chain of $G$ iff $c_1$ is an oriented simple chain of $G$, and

(ii) if $c_1$ is an oriented simple chain of $G$, then $c_1$ is an oriented path of $G$.

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