JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Graphs

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Summary. Definitions of graphs are introduced and their basic properties are proved. The following notions related to graph theory are introduced: subgraph, finite graph, chain and oriented chain - as a finite sequence of edges, path and oriented path - as a finite sequence of different edges, cycle and oriented cycle, incidency of graph's vertices, a sum of two graphs, a degree of a vertice, a set of all subgraphs of a graph. Many ideas of this article have been taken from [11].

MML Identifier: `GRAPH_1`.

WWW: `http://mizar.org/JFM/Vol2/graph_1.html`

The articles [9], [7], [10], [12], [4], [5], [3], [8], [6], [1], and [2] provide the notation and terminology for this paper.

We use the following convention: x, y, v are sets and n, m are natural numbers.

We consider multi graph structures as systems

$\langle$ vertices, edges, a source, a target $\rangle$,

where the vertices and the edges constitute sets and the source and the target are functions from the edges into the vertices.

Let I_1 be a multi graph structure. We say that I_1 is graph-like if and only if:

(Def. 1) The vertices of I_1 are a non empty set.

Let us note that there exists a multi graph structure which is strict and graph-like.

A graph is a graph-like multi graph structure.

In the sequel G, G_1, G_2, G_3 denote graphs.

Let us consider G_1, G_2. Let us assume that the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. The functor $G_1 \cup G_2$ yielding a strict graph is defined by the conditions (Def. 2).

(Def. 2)(i) The vertices of $G_1 \cup G_2 =$ (the vertices of G_1) $\cup$ (the vertices of G_2),

(ii) the edges of $G_1 \cup G_2 =$ (the edges of G_1) $\cup$ (the edges of G_2),

(iii) for every v such that $v \in$ the edges of G_1 holds (the source of $G_1 \cup G_2)(v) =$ (the source of $G_1)(v)$ and (the target of $G_1 \cup G_2)(v) =$ (the target of $G_1)(v)$, and

(iv) for every v such that $v \in$ the edges of G_2 holds (the source of $G_1 \cup G_2)(v) =$ (the source of $G_2)(v)$ and (the target of $G_1 \cup G_2)(v) =$ (the target of $G_2)(v)$.

Let G, G_1, G_2 be graphs. We say that G is a sum of G_1 and G_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) The target of $G_1 \approx$ the target of G_2,

(ii) the source of $G_1 \approx$ the source of G_2, and

(iii) the multi graph structure of $G = G_1 \cup G_2$.

Let I_1 be a graph. We say that I_1 is oriented if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let given x, y. Suppose that

(i) $x \in$ the edges of I_1,

(ii) $y \in$ the edges of I_1,

(iii) (the source of $I_1)(x) =$ (the source of $I_1)(y)$, and

(iv) (the target of $I_1)(x) =$ (the target of $I_1)(y)$.

Then $x = y$.

We say that I_1 is non-multi if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let given x, y. Suppose that

(i) $x \in$ the edges of I_1,

(ii) $y \in$ the edges of I_1, and

(iii) (the source of $I_1)(x) =$ (the source of $I_1)(y)$ and (the target of $I_1)(x) =$ (the target of $I_1)(y)$ or (the source of $I_1)(x) =$ (the target of $I_1)(y)$ and (the source of $I_1)(y) =$ (the target of $I_1)(x)$.

Then $x = y$.

We say that I_1 is simple if and only if:

(Def. 6) It is not true that there exists x such that $x \in$ the edges of I_1 and (the source of $I_1)(x) =$ (the target of $I_1)(x)$.

We say that I_1 is connected if and only if:

(Def. 7) It is not true that there exist graphs G_1, G_2 such that the vertices of G_1 misses the vertices of G_2 and I_1 is a sum of G_1 and G_2.

Let I_1 be a multi graph structure. We say that I_1 is finite if and only if:

(Def. 8) The vertices of I_1 are finite and the edges of I_1 are finite.

One can verify that there exists a multi graph structure which is finite and there exists a graph which is finite, non-multi, oriented, simple, and connected.

In the sequel x, y denote elements of the vertices of G.

Let us consider G, let us consider x, y, and let us consider v. We say that v joins x with y if and only if:

(Def. 9) (The source of $G)(v) = x$ and (the target of $G)(v) = y$ or (the source of $G)(v) = y$ and (the target of $G)(v) = x$.

Let us consider G and let x, y be elements of the vertices of G. We say that x and y are incident if and only if:

(Def. 10) There exists a set v such that $v \in$ the edges of G and v joins x with y.

Let G be a graph. A finite sequence is called a chain of G if it satisfies the conditions (Def. 11).

(Def. 11)(i) For every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n) \in$ the edges of G, and

(ii) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $\mathrm{it}(n)$ joins x' with y'.

Let G be a graph. We see that the chain of G is a finite sequence of elements of the edges of G.

Let G be a graph and let I_1 be a chain of G. We say that I_1 is oriented if and only if:

(Def. 12) For every n such that $1 \leq n$ and $n < \operatorname{len} I_1$ holds (the source of $G)(I_1(n+1)) =$ (the target of $G)(I_1(n))$.

Let G be a graph. Note that there exists a chain of G which is oriented.

Let G be a graph and let I_1 be a chain of G. Let us observe that I_1 is one-to-one if and only if:

(Def. 13) For all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} I_1$ holds $I_1(n) \neq I_1(m)$.

Let G be a graph. One can check that there exists a chain of G which is one-to-one.

Let G be a graph. A path of G is an one-to-one chain of G.

Let G be a graph. Note that there exists a chain of G which is one-to-one and oriented.

Let G be a graph. An oriented path of G is an one-to-one oriented chain of G.

Let G be a graph and let I_1 be a path of G. We say that I_1 is cyclic if and only if the condition (Def. 15) is satisfied.

(Def. 15)[1] There exists a finite sequence p such that

(i) $\operatorname{len} p = \operatorname{len} I_1 + 1$,

(ii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G,

(iii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} I_1$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $I_1(n)$ joins x' with y', and

(iv) $p(1) = p(\operatorname{len} p)$.

Let G be a graph. One can verify that there exists a path of G which is cyclic.

Let G be a graph. A cycle of G is a cyclic path of G.

Let G be a graph. Observe that there exists an oriented path of G which is cyclic.

Let G be a graph. An oriented cycle of G is a cyclic oriented path of G.

Let G be a graph. A graph is called a subgraph of G if it satisfies the conditions (Def. 17).

(Def. 17)[2] (i) The vertices of it $\subseteq$ the vertices of G,

(ii) the edges of it $\subseteq$ the edges of G, and

(iii) for every v such that $v \in$ the edges of it holds (the source of it)$(v) =$ (the source of G)(v) and (the target of it)$(v) =$ (the target of G)(v) and (the source of G)$(v) \in$ the vertices of it and (the target of G)$(v) \in$ the vertices of it.

Let G be a graph. Observe that there exists a subgraph of G which is strict.

Let G be a finite graph. The number of vertices of G yielding a natural number is defined as follows:

(Def. 18) There exists a finite set B such that $B =$ the vertices of G and the number of vertices of $G = \operatorname{card} B$.

The number of edges of G yielding a natural number is defined by:

(Def. 19) There exists a finite set B such that $B =$ the edges of G and the number of edges of $G = \operatorname{card} B$.

Let G be a finite graph and let x be an element of the vertices of G. The functor $\operatorname{EdgIn}(x)$ yielding a natural number is defined by the condition (Def. 20).

(Def. 20) There exists a finite set X such that for every set z holds $z \in X$ iff $z \in$ the edges of G and (the target of G)$(z) = x$ and $\operatorname{EdgIn}(x) = \operatorname{card} X$.

The functor $\operatorname{EdgOut}(x)$ yielding a natural number is defined by the condition (Def. 21).

(Def. 21) There exists a finite set X such that for every set z holds $z \in X$ iff $z \in$ the edges of G and (the source of G)$(z) = x$ and $\operatorname{EdgOut}(x) = \operatorname{card} X$.

Let G be a finite graph and let x be an element of the vertices of G. The degree of x yields a natural number and is defined as follows:

[1] The definition (Def. 14) has been removed.

[2] The definition (Def. 16) has been removed.

(Def. 22) The degree of $x = \mathrm{EdgIn}(x) + \mathrm{EdgOut}(x)$.

Let G_1, G_2 be graphs. The predicate $G_1 \subseteq G_2$ is defined as follows:

(Def. 23) G_1 is a subgraph of G_2.

Let us note that the predicate $G_1 \subseteq G_2$ is reflexive.
Let G be a graph. The functor 2^G yields a set and is defined by:

(Def. 24) For every set x holds $x \in 2^G$ iff x is a strict subgraph of G.

The scheme *GraphSeparation* deals with a graph $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:
There exists a set X such that for every set x holds $x \in X$ iff x is a strict subgraph of $\mathcal{A}$ and $\mathcal{P}[x]$
for all values of the parameters.
One can prove the following propositions:

(1) Let G be a graph. Then
(i) $\mathrm{dom}(\text{the source of } G) = \text{the edges of } G$,
(ii) $\mathrm{dom}(\text{the target of } G) = \text{the edges of } G$,
(iii) $\mathrm{rng}(\text{the source of } G) \subseteq \text{the vertices of } G$, and
(iv) $\mathrm{rng}(\text{the target of } G) \subseteq \text{the vertices of } G$.

(2) For every element x of the vertices of G holds $x \in$ the vertices of G.

(3) Let v be a set. Suppose $v \in$ the edges of G. Then (the source of $G)(v) \in$ the vertices of G and (the target of $G)(v) \in$ the vertices of G.

(4) For every chain p of G holds $p \restriction \mathrm{Seg}\, n$ is a chain of G.

(5) If $G_1 \subseteq G$, then the source of $G_1 \subseteq$ the source of G and the target of $G_1 \subseteq$ the target of G.

(6) Suppose the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. Then
(i) the source of $G_1 \cup G_2 = (\text{the source of } G_1) \cup (\text{the source of } G_2)$, and
(ii) the target of $G_1 \cup G_2 = (\text{the target of } G_1) \cup (\text{the target of } G_2)$.

(7) For every strict graph G holds $G = G \cup G$.

(8) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \cup G_2 = G_2 \cup G_1$.

(9) Suppose that
(i) the source of $G_1 \approx$ the source of G_2,
(ii) the target of $G_1 \approx$ the target of G_2,
(iii) the source of $G_1 \approx$ the source of G_3,
(iv) the target of $G_1 \approx$ the target of G_3,
(v) the source of $G_2 \approx$ the source of G_3, and
(vi) the target of $G_2 \approx$ the target of G_3.
Then $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(10) If G is a sum of G_1 and G_2, then G is a sum of G_2 and G_1.

(11) Every strict graph G is a sum of G and G.

(12) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 = G_2 \cup G_1$.

(13) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ and $G_3 \subseteq G$, then $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(14) $\emptyset$ is a cyclic oriented path of G.

(15) Let H_1, H_2 be strict subgraphs of G. Suppose the vertices of $H_1 =$ the vertices of H_2 and the edges of $H_1 =$ the edges of H_2. Then $H_1 = H_2$.

(16) For all strict graphs G_1, G_2 such that $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$ holds $G_1 = G_2$.

(17) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_3$, then $G_1 \subseteq G_3$.

(18) If G is a sum of G_1 and G_2, then $G_1 \subseteq G$ and $G_2 \subseteq G$.

(19) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(20) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(21) If $G_1 \subseteq G_3$ and $G_2 \subseteq G_3$ and G is a sum of G_1 and G_2, then $G \subseteq G_3$.

(22) If $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 \subseteq G$.

(23) For all strict graphs G_1, G_2 such that $G_1 \subseteq G_2$ holds $G_1 \cup G_2 = G_2$ and $G_2 \cup G_1 = G_2$.

(24) Suppose the source of $G_1 \approx$ the source of G_2 but the target of $G_1 \approx$ the target of G_2 but $G_1 \cup G_2 = G_2$ or $G_2 \cup G_1 = G_2$. Then $G_1 \subseteq G_2$.

(27)[3] For every oriented graph G such that $G_1 \subseteq G$ holds G_1 is oriented.

(28) For every non-multi graph G such that $G_1 \subseteq G$ holds G_1 is non-multi.

(29) For every simple graph G such that $G_1 \subseteq G$ holds G_1 is simple.

(30) For every strict graph G_1 holds $G_1 \in 2^G$ iff $G_1 \subseteq G$.

(31) For every strict graph G holds $G \in 2^G$.

(32) For every strict graph G_1 holds $G_1 \subseteq G_2$ iff $2^{G_1} \subseteq 2^{G_2}$.

(34)[4] For every strict graph G holds $\{G\} \subseteq 2^G$.

(35) Let G_1, G_2 be strict graphs. Suppose the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and $2^{G_1 \cup G_2} \subseteq 2^{G_1} \cup 2^{G_2}$. Then $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$.

(36) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $2^{G_1} \cup 2^{G_2} \subseteq 2^{G_1 \cup G_2}$.

(37) If $G_1 \in 2^G$ and $G_2 \in 2^G$, then $G_1 \cup G_2 \in 2^G$.

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[3] The propositions (25) and (26) have been removed.

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Received December 5, 1990

Published January 2, 2004