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Introduction to Go-Board — Part I

Jarosław Kotowicz
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. In the article we introduce Go-board as some kinds of matrix which elements belong to topological space $\mathcal{E}^2_{\mathrm{T}}$. We define the functor of delaying column in Go-board and relation between Go-board and finite sequence of point from $\mathcal{E}^2_{\mathrm{T}}$. Basic facts about those notations are proved. The concept of the article is based on [16].

MML Identifier: `GOBOARD1`.

WWW: `http://mizar.org/JFM/Vol4/goboard1.html`

The articles [17], [5], [20], [10], [18], [2], [21], [4], [1], [3], [7], [13], [14], [15], [6], [19], [8], [9], [11], and [12] provide the notation and terminology for this paper.

1. Real Numbers Preliminaries

For simplicity, we follow the rules: f, f_1, f_2, g denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, v denotes a finite sequence of elements of $\mathbb{R}$, r, s denote real numbers, n, m, i, j, k denote natural numbers, and x denotes a set.

Next we state three propositions:

(1) $|r-s|=1$ iff $r>s$ and $r=s+1$ or $r<s$ and $s=r+1$.

(2) $|i-j|+|n-m|=1$ iff $|i-j|=1$ and $n=m$ or $|n-m|=1$ and $i=j$.

(3) $n>1$ iff there exists m such that $n=m+1$ and $m>0$.

2. Finite Sequences Preliminaries

The scheme *FinSeqDChoice* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite sequence f of elements of $\mathcal{A}$ such that $\operatorname{len} f=\mathcal{B}$ and for every n such that $n\in \operatorname{Seg}\mathcal{B}$ holds $\mathcal{P}[n,f_n]$

provided the parameters satisfy the following condition:

- For every n such that $n\in\operatorname{Seg}\mathcal{B}$ there exists an element d of $\mathcal{A}$ such that $\mathcal{P}[n,d]$.

We now state several propositions:

(4) If $n=m+1$ and $i\in\operatorname{Seg}n$, then $\operatorname{len}\operatorname{Sgm}(\operatorname{Seg}n\setminus\{i\})=m$.

(5) Suppose $n=m+1$ and $k\in\operatorname{Seg}n$ and $i\in\operatorname{Seg}m$. Then

(i) if $1\leq i$ and $i<k$, then $(\operatorname{Sgm}(\operatorname{Seg}n\setminus\{k\}))(i)=i$, and

(ii) if $k\leq i$ and $i\leq m$, then $(\operatorname{Sgm}(\operatorname{Seg}n\setminus\{k\}))(i)=i+1$.

(6) For every finite sequence f and for all n, m such that $\operatorname{len} f = m+1$ and $n \in \operatorname{dom} f$ holds $\operatorname{len}(f_{\restriction n}) = m$.

(7) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m+1$ and $n \in \operatorname{dom} f$ and $k \in \operatorname{Seg} m$ holds $f_{\restriction n}(k) = f(k)$ or $f_{\restriction n}(k) = f(k+1)$.

(8) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m+1$ and $k < n$ holds $f_{\restriction n}(k) = f(k)$.

(9) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m+1$ and $n \in \operatorname{dom} f$ and $n \leq k$ and $k \leq m$ holds $f_{\restriction n}(k) = f(k+1)$.

(10) Let D be a set, f be a finite sequence of elements of D, and given n, m. If $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$, then $m \in \operatorname{dom} f$ and $(f \restriction n)_m = f_m$.

Let f be a finite sequence of elements of $\mathbb{R}$ and let k be a natural number. Then $f(k)$ is a real number.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is increasing if and only if:

(Def. 1) For all n, m such that $n \in \operatorname{dom} I_1$ and $m \in \operatorname{dom} I_1$ and $n < m$ holds $I_1(n) < I_1(m)$.

Let f be a finite sequence. Let us observe that f is constant if and only if:

(Def. 2) For all n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ holds $f(n) = f(m)$.

Let us note that there exists a finite sequence of elements of $\mathbb{R}$ which is non empty and increasing.

Let D be a non empty set. Note that there exists a finite sequence of elements of D which is non empty.

Let us mention that there exists a finite sequence of elements of $\mathbb{R}$ which is constant.

Let us consider f. The functor $\mathbf{X}$-coordinate(f) yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 3) $\operatorname{len} \mathbf{X}\text{-coordinate}(f) = \operatorname{len} f$ and for every n such that $n \in \operatorname{dom} \mathbf{X}\text{-coordinate}(f)$ holds $(\mathbf{X}\text{-coordinate}(f))(n) = (f_n)_{\mathbf{1}}$.

The functor $\mathbf{Y}$-coordinate(f) yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 4) $\operatorname{len} \mathbf{Y}\text{-coordinate}(f) = \operatorname{len} f$ and for every n such that $n \in \operatorname{dom} \mathbf{Y}\text{-coordinate}(f)$ holds $(\mathbf{Y}\text{-coordinate}(f))(n) = (f_n)_{\mathbf{2}}$.

We now state three propositions:

(14)[1] Suppose that $v \neq \emptyset$ and $\operatorname{rng} v \subseteq \operatorname{Seg} n$ and $v(\operatorname{len} v) = n$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r-s| = 1$ or $r = s$ and $i \in \operatorname{Seg} n$ and $i+1 \in \operatorname{Seg} n$ and $m \in \operatorname{dom} v$ and $v(m) = i$ and for every k such that $k \in \operatorname{dom} v$ and $v(k) = i$ holds $k \leq m$. Then $m+1 \in \operatorname{dom} v$ and $v(m+1) = i+1$.

(15) Suppose that

(i) $v \neq \emptyset$,

(ii) $\operatorname{rng} v \subseteq \operatorname{Seg} n$,

(iii) $v(1) = 1$,

(iv) $v(\operatorname{len} v) = n$, and

(v) for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r-s| = 1$ or $r = s$.

Then

(vi) for every i such that $i \in \operatorname{Seg} n$ there exists k such that $k \in \operatorname{dom} v$ and $v(k) = i$, and

(vii) for all m, k, i, r such that $m \in \operatorname{dom} v$ and $v(m) = i$ and for every j such that $j \in \operatorname{dom} v$ and $v(j) = i$ holds $j \leq m$ and $m < k$ and $k \in \operatorname{dom} v$ and $r = v(k)$ holds $i < r$.

(16) If $i \in \operatorname{dom} f$ and $2 \leq \operatorname{len} f$, then $f_i \in \widetilde{\mathcal{L}}(f)$.

[1] The propositions (11)–(13) have been removed.

3. Matrix Preliminaries

One can prove the following proposition

(17) For every non empty set D and for every matrix M over D and for all i, j such that $j \in \operatorname{dom} M$ and $i \in \operatorname{Seg} \operatorname{width} M$ holds $M_{\square,i}(j) = \operatorname{Line}(M, j)(i)$.

Let D be a non empty set and let M be a matrix over D. Let us observe that M is empty yielding if and only if:

(Def. 5) $0 = \operatorname{len} M$ or $0 = \operatorname{width} M$.

Let M be a matrix over $\mathcal{E}^2_{\mathrm{T}}$. We say that M is line **X**-constant if and only if:

(Def. 6) For every n such that $n \in \operatorname{dom} M$ holds **X**-coordinate$(\operatorname{Line}(M,n))$ is constant.

We say that M is column **Y**-constant if and only if:

(Def. 7) For every n such that $n \in \operatorname{Seg} \operatorname{width} M$ holds **Y**-coordinate$(M_{\square,n})$ is constant.

We say that M is line **Y**-increasing if and only if:

(Def. 8) For every n such that $n \in \operatorname{dom} M$ holds **Y**-coordinate$(\operatorname{Line}(M,n))$ is increasing.

We say that M is column **X**-increasing if and only if:

(Def. 9) For every n such that $n \in \operatorname{Seg} \operatorname{width} M$ holds **X**-coordinate$(M_{\square,n})$ is increasing.

One can check that there exists a matrix over $\mathcal{E}^2_{\mathrm{T}}$ which is non empty yielding, line **X**-constant, column **Y**-constant, line **Y**-increasing, and column **X**-increasing.

Next we state two propositions:

(19)[2] Let M be a column **X**-increasing line **X**-constant matrix over $\mathcal{E}^2_{\mathrm{T}}$ and given x, n, m. If $x \in \operatorname{rng} \operatorname{Line}(M,n)$ and $x \in \operatorname{rng} \operatorname{Line}(M,m)$ and $n \in \operatorname{dom} M$ and $m \in \operatorname{dom} M$, then $n = m$.

(20) Let M be a line **Y**-increasing column **Y**-constant matrix over $\mathcal{E}^2_{\mathrm{T}}$ and given x, n, m. If $x \in \operatorname{rng}(M_{\square,n})$ and $x \in \operatorname{rng}(M_{\square,m})$ and $n \in \operatorname{Seg} \operatorname{width} M$ and $m \in \operatorname{Seg} \operatorname{width} M$, then $n = m$.

4. Basic Go-Board‘s Notation

A Go-board is a non empty yielding line **X**-constant column **Y**-constant line **Y**-increasing column **X**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$.

In the sequel G is a Go-board.

One can prove the following propositions:

(21) If $x = G \circ (m,k)$ and $x = G \circ (i,j)$ and $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G, then $m = i$ and $k = j$.

(22) If $m \in \operatorname{dom} f$ and $f_1 \in \operatorname{rng}(G_{\square,1})$, then $(f{\restriction}m)_1 \in \operatorname{rng}(G_{\square,1})$.

(23) If $m \in \operatorname{dom} f$ and $f_m \in \operatorname{rng}(G_{\square,\operatorname{width} G})$, then $(f{\restriction}m)_{\operatorname{len}(f{\restriction}m)} \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.

(24) If $\operatorname{rng} f$ misses $\operatorname{rng}(G_{\square,i})$ and $f_n = G \circ (m,k)$ and $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} G$, then $i \neq k$.

Let us consider G, i. Let us assume that $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$. The deleting of i-column in G yielding a Go-board is defined by the conditions (Def. 10).

(Def. 10)(i) len(the deleting of i-column in G) $= \operatorname{len} G$, and

(ii) for every k such that $k \in \operatorname{dom} G$ holds (the deleting of i-column in G)$(k) = \operatorname{Line}(G,k)_{\restriction i}$.

One can prove the following propositions:

[2] The proposition (18) has been removed.

(25) If $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G > 1$ and $k \in \operatorname{dom} G$, then Line(the deleting of i-column in G, k) $= \operatorname{Line}(G,k)_{\restriction i}$.

(26) If $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$, then width(the deleting of i-column in G) $= m$.

(27) If $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G > 1$, then $\operatorname{width} G =$ width(the deleting of i-column in G) $+ 1$.

(28) Suppose $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G > 1$ and $n \in \operatorname{dom} G$ and $m \in \operatorname{Seg}$ width(the deleting of i-column in G). Then (the deleting of i-column in G) $\circ (n,m) = \operatorname{Line}(G,n)_{\restriction i}(m)$.

(29) Suppose $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $1 \leq k$ and $k < i$. Then (the deleting of i-column in $G)_{\square,k} = G_{\square,k}$ and $k \in \operatorname{Seg}$ width(the deleting of i-column in G) and $k \in \operatorname{Seg}\operatorname{width} G$.

(30) Suppose $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in \operatorname{Seg}$ width(the deleting of i-column in G) and $k+1 \in \operatorname{Seg}\operatorname{width} G$.

(31) Suppose $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $n \in \operatorname{dom} G$ and $1 \leq k$ and $k < i$. Then (the deleting of i-column in G) $\circ (n,k) = G \circ (n,k)$ and $k \in \operatorname{Seg}\operatorname{width} G$.

(32) Suppose $i \in \operatorname{Seg}\operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $n \in \operatorname{dom} G$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in G) $\circ (n,k) = G \circ (n,k+1)$ and $k+1 \in \operatorname{Seg}\operatorname{width} G$.

(33) Suppose $\operatorname{width} G = m+1$ and $m > 0$ and $k \in \operatorname{Seg} m$. Then (the deleting of 1-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in \operatorname{Seg}$ width(the deleting of 1-column in G) and $k+1 \in \operatorname{Seg}\operatorname{width} G$.

(34) If $\operatorname{width} G = m+1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{dom} G$, then (the deleting of 1-column in G) $\circ (n,k) = G \circ (n,k+1)$ and $1 \in \operatorname{Seg}\operatorname{width} G$.

(35) Suppose $\operatorname{width} G = m+1$ and $m > 0$ and $k \in \operatorname{Seg} m$. Then (the deleting of width G-column in $G)_{\square,k} = G_{\square,k}$ and $k \in \operatorname{Seg}$ width(the deleting of width G-column in G).

(36) If $\operatorname{width} G = m+1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{dom} G$, then $k \in \operatorname{Seg}\operatorname{width} G$ and (the deleting of width G-column in G) $\circ (n,k) = G \circ (n,k)$ and $\operatorname{width} G \in \operatorname{Seg}\operatorname{width} G$.

(37) Suppose $\operatorname{rng} f$ misses $\operatorname{rng}(G_{\square,i})$ and $f_n \in \operatorname{rng}\operatorname{Line}(G,m)$ and $n \in \operatorname{dom} f$ and $i \in \operatorname{Seg}\operatorname{width} G$ and $m \in \operatorname{dom} G$ and $\operatorname{width} G > 1$. Then $f_n \in \operatorname{rng}$ Line(the deleting of i-column in G, m).

In the sequel D is a set, f is a finite sequence of elements of D, and M is a matrix over D.

Let us consider D, f, M. We say that f is a sequence which elements belong to M if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) For every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of M and $f_n = M \circ (i,j)$, and

(ii) for every n such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and for all m, k, i, j such that $\langle m, k\rangle \in$ the indices of M and $\langle i, j\rangle \in$ the indices of M and $f_n = M \circ (m,k)$ and $f_{n+1} = M \circ (i,j)$ holds $|m-i| + |k-j| = 1$.

We now state three propositions:

(38)(i) If $m \in \operatorname{dom} f$, then $1 \leq \operatorname{len}(f \restriction m)$, and

(ii) if f is a sequence which elements belong to M, then $f \restriction m$ is a sequence which elements belong to M.

(39) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $(f_1)_n = M \circ (i, j)$, and

(ii) for every n such that $n \in \operatorname{dom} f_2$ there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $(f_2)_n = M \circ (i, j)$.

Let given n. If $n \in \operatorname{dom}(f_1 \frown f_2)$, then there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $(f_1 \frown f_2)_n = M \circ (i, j)$.

(40) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ and for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of M and $\langle i, j \rangle \in$ the indices of M and $(f_1)_n = M \circ (m, k)$ and $(f_1)_{n+1} = M \circ (i, j)$ holds $|m - i| + |k - j| = 1$,

(ii) for every n such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ and for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of M and $\langle i, j \rangle \in$ the indices of M and $(f_2)_n = M \circ (m, k)$ and $(f_2)_{n+1} = M \circ (i, j)$ holds $|m - i| + |k - j| = 1$, and

(iii) for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of M and $\langle i, j \rangle \in$ the indices of M and $(f_1)_{\operatorname{len} f_1} = M \circ (m, k)$ and $(f_2)_1 = M \circ (i, j)$ and $\operatorname{len} f_1 \in \operatorname{dom} f_1$ and $1 \in \operatorname{dom} f_2$ holds $|m - i| + |k - j| = 1$.

Let given n. Suppose $n \in \operatorname{dom}(f_1 \frown f_2)$ and $n+1 \in \operatorname{dom}(f_1 \frown f_2)$. Let given m, k, i, j. Suppose $\langle m, k \rangle \in$ the indices of M and $\langle i, j \rangle \in$ the indices of M and $(f_1 \frown f_2)_n = M \circ (m, k)$ and $(f_1 \frown f_2)_{n+1} = M \circ (i, j)$. Then $|m - i| + |k - j| = 1$.

In the sequel f denotes a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.
Next we state a number of propositions:

(41) Suppose f is a sequence which elements belong to G and $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{rng} f$ misses $\operatorname{rng}(G_{\square,i})$ and $\operatorname{width} G > 1$. Then f is a sequence which elements belong to the deleting of i-column in G.

(42) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{dom} G$ and $f_i \in \operatorname{rng} \operatorname{Line}(G, n)$.

(43) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $n \in \operatorname{dom} G$ and $f_i \in \operatorname{rng} \operatorname{Line}(G, n)$. Then $f_{i+1} \in \operatorname{rng} \operatorname{Line}(G, n)$ or for every k such that $f_{i+1} \in \operatorname{rng} \operatorname{Line}(G, k)$ and $k \in \operatorname{dom} G$ holds $|n - k| = 1$.

(44) Suppose that $1 \leq \operatorname{len} f$ and $f_{\operatorname{len} f} \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$ and f is a sequence which elements belong to G and $i \in \operatorname{dom} G$ and $i+1 \in \operatorname{dom} G$ and $m \in \operatorname{dom} f$ and $f_m \in \operatorname{rng} \operatorname{Line}(G, i)$ and for every k such that $k \in \operatorname{dom} f$ and $f_k \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $k \leq m$. Then $m+1 \in \operatorname{dom} f$ and $f_{m+1} \in \operatorname{rng} \operatorname{Line}(G, i+1)$.

(45) Suppose $1 \leq \operatorname{len} f$ and $f_1 \in \operatorname{rng} \operatorname{Line}(G, 1)$ and $f_{\operatorname{len} f} \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$ and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ there exists k such that $k \in \operatorname{dom} f$ and $f_k \in \operatorname{rng} \operatorname{Line}(G, i)$,

(ii) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $2 \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(f)$ meets $\operatorname{rng} \operatorname{Line}(G, i)$, and

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{len} G$ and $k \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $f_k \in \operatorname{rng} \operatorname{Line}(G, i)$ and for every n such that $n \in \operatorname{dom} f$ and $f_n \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $n \leq k$ and $k < m$ and $f_m \in \operatorname{rng} \operatorname{Line}(G, j)$ holds $i < j$.

(46) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{Seg} \operatorname{width} G$ and $f_i \in \operatorname{rng}(G_{\square,n})$.

(47) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $n \in \operatorname{Seg} \operatorname{width} G$ and $f_i \in \operatorname{rng}(G_{\square,n})$. Then $f_{i+1} \in \operatorname{rng}(G_{\square,n})$ or for every k such that $f_{i+1} \in \operatorname{rng}(G_{\square,k})$ and $k \in \operatorname{Seg} \operatorname{width} G$ holds $|n - k| = 1$.

(48) Suppose that $1 \leq \mathrm{len} f$ and $f_{\mathrm{len} f} \in \mathrm{rng}(G_{\square,\mathrm{width} G})$ and f is a sequence which elements belong to G and $i \in \mathrm{Seg}\,\mathrm{width} G$ and $i+1 \in \mathrm{Seg}\,\mathrm{width} G$ and $m \in \mathrm{dom} f$ and $f_m \in \mathrm{rng}(G_{\square,i})$ and for every k such that $k \in \mathrm{dom} f$ and $f_k \in \mathrm{rng}(G_{\square,i})$ holds $k \leq m$. Then $m+1 \in \mathrm{dom} f$ and $f_{m+1} \in \mathrm{rng}(G_{\square,i+1})$.

(49) Suppose $1 \leq \mathrm{len} f$ and $f_1 \in \mathrm{rng}(G_{\square,1})$ and $f_{\mathrm{len} f} \in \mathrm{rng}(G_{\square,\mathrm{width} G})$ and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \mathrm{width} G$ there exists k such that $k \in \mathrm{dom} f$ and $f_k \in \mathrm{rng}(G_{\square,i})$,

(ii) for every i such that $1 \leq i$ and $i \leq \mathrm{width} G$ and $2 \leq \mathrm{len} f$ holds $\widetilde{\mathcal{L}}(f)$ meets $\mathrm{rng}(G_{\square,i})$, and

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \mathrm{width} G$ and $1 \leq j$ and $j \leq \mathrm{width} G$ and $k \in \mathrm{dom} f$ and $m \in \mathrm{dom} f$ and $f_k \in \mathrm{rng}(G_{\square,i})$ and for every n such that $n \in \mathrm{dom} f$ and $f_n \in \mathrm{rng}(G_{\square,i})$ holds $n \leq k$ and $k < m$ and $f_m \in \mathrm{rng}(G_{\square,j})$ holds $i < j$.

(50) Suppose that

(i) $n \in \mathrm{dom} f$,

(ii) $f_n \in \mathrm{rng}(G_{\square,k})$,

(iii) $k \in \mathrm{Seg}\,\mathrm{width} G$,

(iv) $f_1 \in \mathrm{rng}(G_{\square,1})$,

(v) f is a sequence which elements belong to G, and

(vi) for every i such that $i \in \mathrm{dom} f$ and $f_i \in \mathrm{rng}(G_{\square,k})$ holds $n \leq i$.

Let given i. If $i \in \mathrm{dom} f$ and $i \leq n$, then for every m such that $m \in \mathrm{Seg}\,\mathrm{width} G$ and $f_i \in \mathrm{rng}(G_{\square,m})$ holds $m \leq k$.

(51) Suppose f is a sequence which elements belong to G and $f_1 \in \mathrm{rng}(G_{\square,1})$ and $f_{\mathrm{len} f} \in \mathrm{rng}(G_{\square,\mathrm{width} G})$ and $\mathrm{width} G > 1$ and $1 \leq \mathrm{len} f$. Then there exists g such that

(i) $g_1 \in \mathrm{rng}((\text{the deleting of width} G\text{-column in } G)_{\square,1})$,

(ii) $g_{\mathrm{len} g} \in \mathrm{rng}((\text{the deleting of width} G\text{-column in } G)_{\square,\mathrm{width}(\text{the deleting of width} G\text{-column in } G)})$,

(iii) $1 \leq \mathrm{len} g$,

(iv) g is a sequence which elements belong to the deleting of width$G$-column in G, and

(v) $\mathrm{rng}\, g \subseteq \mathrm{rng} f$.

(52) Suppose f is a sequence which elements belong to G and $\mathrm{rng} f \cap \mathrm{rng}(G_{\square,1}) \neq \emptyset$ and $\mathrm{rng} f \cap \mathrm{rng}(G_{\square,\mathrm{width} G}) \neq \emptyset$. Then there exists g such that $\mathrm{rng}\, g \subseteq \mathrm{rng} f$ and $g_1 \in \mathrm{rng}(G_{\square,1})$ and $g_{\mathrm{len} g} \in \mathrm{rng}(G_{\square,\mathrm{width} G})$ and $1 \leq \mathrm{len} g$ and g is a sequence which elements belong to G.

(53) Suppose $k \in \mathrm{dom} G$ and f is a sequence which elements belong to G and $f_{\mathrm{len} f} \in \mathrm{rng}\,\mathrm{Line}(G, \mathrm{len} G)$ and $n \in \mathrm{dom} f$ and $f_n \in \mathrm{rng}\,\mathrm{Line}(G,k)$. Then

(i) for every i such that $k \leq i$ and $i \leq \mathrm{len} G$ there exists j such that $j \in \mathrm{dom} f$ and $n \leq j$ and $f_j \in \mathrm{rng}\,\mathrm{Line}(G,i)$, and

(ii) for every i such that $k < i$ and $i \leq \mathrm{len} G$ there exists j such that $j \in \mathrm{dom} f$ and $n < j$ and $f_j \in \mathrm{rng}\,\mathrm{Line}(G,i)$.

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