JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Concept of Fuzzy Set and Membership Function and Basic Properties of Fuzzy Set Operation

Takashi Mitsuishi
Shinshu University
Nagano

Noboru Endou
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. This article introduces the fuzzy theory. At first, the definition of fuzzy set characterized by membership function is described. Next, definitions of empty fuzzy set and universal fuzzy set and basic operations of these fuzzy sets are shown. At last, exclusive sum and absolute difference which are special operation are introduced.

MML Identifier: FUZZY_1.

WWW: http://mizar.org/JFM/Vol12/fuzzy_1.html

The articles [7], [1], [10], [8], [9], [2], [5], [11], [3], [4], and [6] provide the notation and terminology for this paper.

1. Definition of Membership Function and Fuzzy Set

In this paper x, y are sets, C is a non empty set, and c is an element of C.

The following proposition is true

(1) $\mathrm{rng}(\chi_{x,y}) \subseteq [0,1]$.

Let us consider C. A partial function from C to $\mathbb{R}$ is said to be a membership function of C if:

(Def. 1) $\mathrm{dom}\,\mathrm{it} = C$ and $\mathrm{rng}\,\mathrm{it} \subseteq [0,1]$.

Next we state the proposition

(2) $\chi_{C,C}$ is a membership function of C.

In the sequel f, h, g, h_1 denote membership functions of C.

Let C be a non empty set and let h be a membership function of C. A set is called a FuzzySet of C, h if:

(Def. 2) $\mathrm{It} = [\!:C, h^\circ C:\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The predicate $A = B$ is defined as follows:

(Def. 3) $h = g$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The predicate $A \subseteq B$ is defined by:

(Def. 4) For every element c of C holds $h(c) \leq g(c)$.

In the sequel A is a FuzzySet of C, f, B is a FuzzySet of C, g, D is a FuzzySet of C, h, and D_1 is a FuzzySet of C, h_1.

One can prove the following propositions:

(3) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(4) $A \subseteq A$.

(5) If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq D$.

2. Intersection, Union and Complement

Let C be a non empty set and let h, g be membership functions of C. The functor $\min(h,g)$ yields a membership function of C and is defined as follows:

(Def. 5) For every element c of C holds $(\min(h,g))(c) = \min(h(c),g(c))$.

Let C be a non empty set and let h, g be membership functions of C. The functor $\max(h,g)$ yields a membership function of C and is defined by:

(Def. 6) For every element c of C holds $(\max(h,g))(c) = \max(h(c),g(c))$.

Let C be a non empty set and let h be a membership function of C. The functor 1-minus$h$ yields a membership function of C and is defined as follows:

(Def. 7) For every element c of C holds $(\text{1-minus}\,h)(c) = 1 - h(c)$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cap B$ yielding a FuzzySet of C, $\min(h,g)$ is defined as follows:

(Def. 8) $A \cap B = [\!: C, (\min(h,g))^\circ C :\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cup B$ yielding a FuzzySet of C, $\max(h,g)$ is defined as follows:

(Def. 9) $A \cup B = [\!: C, (\max(h,g))^\circ C :\!]$.

Let C be a non empty set, let h be a membership function of C, and let A be a FuzzySet of C, h. The functor $A^{\rm c}$ yields a FuzzySet of C, 1-minus$h$ and is defined as follows:

(Def. 10) $A^{\rm c} = [\!: C, (\text{1-minus}\,h)^\circ C :\!]$.

The following propositions are true:

(6) $\min(h(c),g(c)) = (\min(h,g))(c)$ and $\max(h(c),g(c)) = (\max(h,g))(c)$.

(7) $\max(h,h) = h$ and $\min(h,h) = h$ and $\max(h,h) = \min(h,h)$ and $\min(f,g) = \min(g,f)$ and $\max(f,g) = \max(g,f)$.

(9)[1] $A \cap A = A$ and $A \cup A = A$.

(10) $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

(11) $\max(\max(f,g),h) = \max(f,\max(g,h))$ and $\min(\min(f,g),h) = \min(f,\min(g,h))$.

(12) $(A \cup B) \cup D = A \cup (B \cup D)$.

(13) $(A \cap B) \cap D = A \cap (B \cap D)$.

(14) $\max(f,\min(f,g)) = f$ and $\min(f,\max(f,g)) = f$.

[1] The proposition (8) has been removed.

(15) $A \cup A \cap B = A$ and $A \cap (A \cup B) = A$.

(16) $\min(f, \max(g,h)) = \max(\min(f,g), \min(f,h))$ and $\max(f, \min(g,h)) = \min(\max(f,g), \max(f,h))$.

(17) $A \cup B \cap D = (A \cup B) \cap (A \cup D)$ and $A \cap (B \cup D) = A \cap B \cup A \cap D$.

(18) $1\text{-minus}\, 1\text{-minus}\, h = h$.

(19) $(A^{\mathrm{c}})^{\mathrm{c}} = A$.

(20) $1\text{-minus} \max(f,g) = \min(1\text{-minus}\, f, 1\text{-minus}\, g)$ and $1\text{-minus} \min(f,g) = \max(1\text{-minus}\, f, 1\text{-minus}\, g)$.

(21) $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$ and $(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}$.

3. Empty Fuzzy Set and Universal Fuzzy Set

Let C be a non empty set. A set is called an Empty FuzzySet of C if:

(Def. 11) It $= [\!: C, (\chi_{\emptyset,C})^{\circ} C :\!]$.

A set is called a Universal FuzzySet of C if:

(Def. 12) It $= [\!: C, (\chi_{C,C})^{\circ} C :\!]$.

In the sequel X denotes a Universal FuzzySet of C and E denotes an Empty FuzzySet of C.
The following proposition is true

(23)[2] $\chi_{\emptyset,C}$ is a membership function of C.

Let C be a non empty set. The functor EMFC yielding a membership function of C is defined as follows:

(Def. 13) $\mathrm{EMF}C = \chi_{\emptyset,C}$.

Let C be a non empty set. The functor UMFC yielding a membership function of C is defined as follows:

(Def. 14) $\mathrm{UMF}C = \chi_{C,C}$.

We now state two propositions:

(26)[3] E is a FuzzySet of C, EMFC.

(27) X is a FuzzySet of C, UMFC.

Let C be a non empty set. We see that the Empty FuzzySet of C is a FuzzySet of C, EMFC. We see that the Universal FuzzySet of C is a FuzzySet of C, UMFC.
In the sequel X is a Universal FuzzySet of C and E is an Empty FuzzySet of C.
The following propositions are true:

(28) Let a, b be elements of $\mathbb{R}$ and f be a partial function from C to $\mathbb{R}$. Suppose $\mathrm{rng}\, f \subseteq [a,b]$ and $a \leq b$. Let x be an element of C. If $x \in \mathrm{dom}\, f$, then $a \leq f(x)$ and $f(x) \leq b$.

(29) $E \subseteq A$.

(30) $A \subseteq X$.

(31) For every element x of C and for every membership function h of C holds $(\mathrm{EMF}C)(x) \leq h(x)$ and $h(x) \leq (\mathrm{UMF}C)(x)$.

[2] The proposition (22) has been removed.
[3] The propositions (24) and (25) have been removed.

(32) $\max(f, \mathrm{UMF}C) = \mathrm{UMF}C$ and $\min(f, \mathrm{UMF}C) = f$ and $\max(f, \mathrm{EMF}C) = f$ and $\min(f, \mathrm{EMF}C) = \mathrm{EMF}C$.

(33) $A \cup X = X$ and $A \cap X = A$.

(34) $A \cup E = A$ and $A \cap E = E$.

(35) $A \subseteq A \cup B$.

(36) If $A \subseteq D$ and $B \subseteq D$, then $A \cup B \subseteq D$.

(38)[4] If $A \subseteq B$, then $A \cup D \subseteq B \cup D$.

(39) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cup D \subseteq B \cup D_1$.

(40) If $A \subseteq B$, then $A \cup B = B$.

(41) $A \cap B \subseteq A$.

(42) $A \cap B \subseteq A \cup B$.

(43) If $D \subseteq A$ and $D \subseteq B$, then $D \subseteq A \cap B$.

(44) For all elements a, b, c, d of $\mathbb{R}$ such that $a \leq b$ and $c \leq d$ holds $\min(a, c) \leq \min(b, d)$.

(45) For all elements a, b, c of $\mathbb{R}$ such that $a \leq b$ holds $\min(a, c) \leq \min(b, c)$.

(46) If $A \subseteq B$, then $A \cap D \subseteq B \cap D$.

(47) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cap D \subseteq B \cap D_1$.

(48) If $A \subseteq B$, then $A \cap B = A$.

(49) If $A \subseteq B$ and $A \subseteq D$ and $B \cap D = E$, then $A = E$.

(50) If $A \cap B \cup A \cap D = A$, then $A \subseteq B \cup D$.

(51) If $A \subseteq B$ and $B \cap D = E$, then $A \cap D = E$.

(52) If $A \subseteq E$, then $A = E$.

(53) $A \cup B = E$ iff $A = E$ and $B = E$.

(54) $A = B \cup D$ iff $B \subseteq A$ and $D \subseteq A$ and for all h_1, D_1 such that $B \subseteq D_1$ and $D \subseteq D_1$ holds $A \subseteq D_1$.

(55) $A = B \cap D$ iff $A \subseteq B$ and $A \subseteq D$ and for all h_1, D_1 such that $D_1 \subseteq B$ and $D_1 \subseteq D$ holds $D_1 \subseteq A$.

(56) If $A \subseteq B \cup D$ and $A \cap D = E$, then $A \subseteq B$.

(57) $A \subseteq B$ iff $B^{\mathrm{c}} \subseteq A^{\mathrm{c}}$.

(58) If $A \subseteq B^{\mathrm{c}}$, then $B \subseteq A^{\mathrm{c}}$.

(59) If $A^{\mathrm{c}} \subseteq B$, then $B^{\mathrm{c}} \subseteq A$.

(60) $(A \cup B)^{\mathrm{c}} \subseteq A^{\mathrm{c}}$ and $(A \cup B)^{\mathrm{c}} \subseteq B^{\mathrm{c}}$.

(61) $A^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}}$ and $B^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}}$.

(62) 1-minus $\mathrm{EMF}C = \mathrm{UMF}C$ and 1-minus $\mathrm{UMF}C = \mathrm{EMF}C$.

(63) $E^{\mathrm{c}} = X$ and $X^{\mathrm{c}} = E$.

[4] The proposition (37) has been removed.

4. Exclusive Sum, Absolute Difference

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \dot{-} B$ yielding a FuzzySet of C, $\max(\min(h, 1\text{-minus}\, g), \min(1\text{-minus}\, h, g))$ is defined as follows:

(Def. 15) $A \dot{-} B = [:C, (\max(\min(h, 1\text{-minus}\, g), \min(1\text{-minus}\, h, g)))^{\circ}C:]$.

The following propositions are true:

(65)[5] $A \dot{-} B = B \dot{-} A$.

(66) $A \dot{-} E = A$ and $E \dot{-} A = A$.

(67) $A \dot{-} X = A^{\rm c}$ and $X \dot{-} A = A^{\rm c}$.

(68) $A \cap B \cup B \cap D \cup D \cap A = (A \cup B) \cap (B \cup D) \cap (D \cup A)$.

(69) $A \cap B \cup A^{\rm c} \cap B^{\rm c} \subseteq (A \dot{-} B)^{\rm c}$.

(70) $(A \dot{-} B) \cup A \cap B \subseteq A \cup B$.

(71) $A \dot{-} A = A \cap A^{\rm c}$.

Let C be a non empty set and let h, g be membership functions of C. The functor $|h - g|$ yielding a membership function of C is defined as follows:

(Def. 16) For every element c of C holds $|h - g|(c) = |h(c) - g(c)|$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $|A - B|$ yielding a FuzzySet of C, $|h - g|$ is defined as follows:

(Def. 17) $|A - B| = [:C, |h - g|^{\circ}C:]$.

References

[1] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[3] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[4] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[5] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[6] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[8] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[9] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[5] The proposition (64) has been removed.

[11] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received May 18, 2000

Published January 2, 2004