JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Basic Functions and Operations on Functions

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Summary. We define the following mappings: the characteristic function of a subset of a set, the inclusion function (injection or embedding), the projections from a Cartesian product onto its arguments and diagonal function (inclusion of a set into its Cartesian square). Some operations on functions are also defined: the products of two functions (the complex function and the more general product-function), the function induced on power sets by the image and inverse-image. Some simple propositions related to the introduced notions are proved.

MML Identifier: `FUNCT_3`.

WWW: `http://mizar.org/JFM/Vol1/funct_3.html`

The articles [4], [3], [5], [6], [7], [1], and [2] provide the notation and terminology for this paper.

We use the following convention: x, y, z, z_1, z_2 denote sets, A, B, V, X, X_1, X_2, Y, Y_1, Y_2, Z denote sets, and C, C_1, C_2, D, D_1, D_2 denote non empty sets.

One can prove the following propositions:

(1) If $A \subseteq Y$, then $\mathrm{id}_A = \mathrm{id}_Y \restriction A$.

(2) For all functions f, g such that $X \subseteq \mathrm{dom}(g \cdot f)$ holds $f^\circ X \subseteq \mathrm{dom}\, g$.

(3) For all functions f, g such that $X \subseteq \mathrm{dom}\, f$ and $f^\circ X \subseteq \mathrm{dom}\, g$ holds $X \subseteq \mathrm{dom}(g \cdot f)$.

(4) For all functions f, g such that $Y \subseteq \mathrm{rng}(g \cdot f)$ and g is one-to-one holds $g^{-1}(Y) \subseteq \mathrm{rng}\, f$.

(5) For all functions f, g such that $Y \subseteq \mathrm{rng}\, g$ and $g^{-1}(Y) \subseteq \mathrm{rng}\, f$ holds $Y \subseteq \mathrm{rng}(g \cdot f)$.

In this article we present several logical schemes. The scheme *FuncEx 3* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\mathrm{dom}\, f = [: \mathcal{A}, \mathcal{B} :]$ and for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds $\mathcal{P}[x, y, f(\langle x, y \rangle)]$

provided the parameters satisfy the following conditions:

- For all x, y, z_1, z_2 such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y, z_1]$ and $\mathcal{P}[x, y, z_2]$ holds $z_1 = z_2$, and
- For all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ there exists z such that $\mathcal{P}[x, y, z]$.

The scheme *Lambda 3* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\mathrm{dom}\, f = [: \mathcal{A}, \mathcal{B} :]$ and for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds $f(\langle x, y \rangle) = \mathcal{F}(x, y)$

for all values of the parameters.

The following proposition is true

(6) Let f, g be functions. Suppose $\operatorname{dom} f = [:X, Y:]$ and $\operatorname{dom} g = [:X, Y:]$ and for all x, y such that $x \in X$ and $y \in Y$ holds $f(\langle x, y\rangle) = g(\langle x, y\rangle)$. Then $f = g$.

Let f be a function. The functor ${}^\circ f$ yields a function and is defined as follows:

(Def. 1) $\operatorname{dom} {}^\circ f = 2^{\operatorname{dom} f}$ and for every X such that $X \subseteq \operatorname{dom} f$ holds $({}^\circ f)(X) = f^\circ X$.

Next we state a number of propositions:

(8)[1] For every function f such that $X \in \operatorname{dom} {}^\circ f$ holds $({}^\circ f)(X) = f^\circ X$.

(9) For every function f holds $({}^\circ f)(\emptyset) = \emptyset$.

(10) For every function f holds $\operatorname{rng} {}^\circ f \subseteq 2^{\operatorname{rng} f}$.

(12)[2] For every function f holds $({}^\circ f)^\circ A \subseteq 2^{\operatorname{rng} f}$.

(13) For every function f holds $({}^\circ f)^{-1}(B) \subseteq 2^{\operatorname{dom} f}$.

(14) For every function f from X into D holds $({}^\circ f)^{-1}(B) \subseteq 2^X$.

(15) For every function f holds $\bigcup(({}^\circ f)^\circ A) \subseteq f^\circ \bigcup A$.

(16) For every function f such that $A \subseteq 2^{\operatorname{dom} f}$ holds $f^\circ \bigcup A = \bigcup(({}^\circ f)^\circ A)$.

(17) For every function f from X into D such that $A \subseteq 2^X$ holds $f^\circ \bigcup A = \bigcup(({}^\circ f)^\circ A)$.

(18) For every function f holds $\bigcup(({}^\circ f)^{-1}(B)) \subseteq f^{-1}(\bigcup B)$.

(19) For every function f such that $B \subseteq 2^{\operatorname{rng} f}$ holds $f^{-1}(\bigcup B) = \bigcup(({}^\circ f)^{-1}(B))$.

(20) For all functions f, g holds ${}^\circ(g \cdot f) = {}^\circ g \cdot {}^\circ f$.

(21) For every function f holds ${}^\circ f$ is a function from $2^{\operatorname{dom} f}$ into $2^{\operatorname{rng} f}$.

(22) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds ${}^\circ f$ is a function from 2^X into 2^Y.

Let us consider X, D and let f be a function from X into D. Then ${}^\circ f$ is a function from 2^X into 2^D.

Let f be a function. The functor ${}^{-1} f$ yields a function and is defined as follows:

(Def. 2) $\operatorname{dom} {}^{-1} f = 2^{\operatorname{rng} f}$ and for every Y such that $Y \subseteq \operatorname{rng} f$ holds $({}^{-1} f)(Y) = f^{-1}(Y)$.

One can prove the following propositions:

(24)[3] For every function f such that $Y \in \operatorname{dom} {}^{-1} f$ holds $({}^{-1} f)(Y) = f^{-1}(Y)$.

(25) For every function f holds $\operatorname{rng} {}^{-1} f \subseteq 2^{\operatorname{dom} f}$.

(27)[4] For every function f holds $({}^{-1} f)^\circ B \subseteq 2^{\operatorname{dom} f}$.

(28) For every function f holds $({}^{-1} f)^{-1}(A) \subseteq 2^{\operatorname{rng} f}$.

(29) For every function f holds $\bigcup(({}^{-1} f)^\circ B) \subseteq f^{-1}(\bigcup B)$.

(30) For every function f such that $B \subseteq 2^{\operatorname{rng} f}$ holds $\bigcup(({}^{-1} f)^\circ B) = f^{-1}(\bigcup B)$.

(31) For every function f holds $\bigcup(({}^{-1} f)^{-1}(A)) \subseteq f^\circ \bigcup A$.

[1] The proposition (7) has been removed.
[2] The proposition (11) has been removed.
[3] The proposition (23) has been removed.
[4] The proposition (26) has been removed.

(32) For every function f such that $A \subseteq 2^{\mathrm{dom}\, f}$ and f is one-to-one holds $\bigcup((^{-1}f)^{-1}(A)) = f^{\circ}\bigcup A$.

(33) For every function f holds $(^{-1}f)^{\circ}B \subseteq (^{\circ}f)^{-1}(B)$.

(34) For every function f such that f is one-to-one holds $(^{-1}f)^{\circ}B = (^{\circ}f)^{-1}(B)$.

(35) For every function f and for every set A such that $A \subseteq 2^{\mathrm{dom}\, f}$ holds $(^{-1}f)^{-1}(A) \subseteq (^{\circ}f)^{\circ}A$.

(36) For every function f and for every set A such that f is one-to-one holds $(^{\circ}f)^{\circ}A \subseteq (^{-1}f)^{-1}(A)$.

(37) For every function f and for every set A such that f is one-to-one and $A \subseteq 2^{\mathrm{dom}\, f}$ holds $(^{-1}f)^{-1}(A) = (^{\circ}f)^{\circ}A$.

(38) For all functions f, g such that g is one-to-one holds $^{-1}(g \cdot f) = {}^{-1}f \cdot {}^{-1}g$.

(39) For every function f holds ^{-1}f is a function from $2^{\mathrm{rng}\, f}$ into $2^{\mathrm{dom}\, f}$.

Let us consider A, X. The functor $\chi_{A,X}$ yields a function and is defined as follows:

(Def. 3) $\mathrm{dom}(\chi_{A,X}) = X$ and for every x such that $x \in X$ holds if $x \in A$, then $\chi_{A,X}(x) = 1$ and if $x \notin A$, then $\chi_{A,X}(x) = 0$.

We now state several propositions:

(42)[5] If $\chi_{A,X}(x) = 1$, then $x \in A$.

(43) If $x \in X \setminus A$, then $\chi_{A,X}(x) = 0$.

(47)[6] If $A \subseteq X$ and $B \subseteq X$ and $\chi_{A,X} = \chi_{B,X}$, then $A = B$.

(48) $\mathrm{rng}(\chi_{A,X}) \subseteq \{0,1\}$.

(49) For every function f from X into $\{0,1\}$ holds $f = \chi_{f^{-1}(\{1\}),X}$.

Let us consider A, X. Then $\chi_{A,X}$ is a function from X into $\{0,1\}$.

Let us consider Y and let A be a subset of Y. The functor $\overset{A}{\hookrightarrow}$ yields a function from A into Y and is defined by:

(Def. 4) $\overset{A}{\hookrightarrow} = \mathrm{id}_A$.

The following four propositions are true:

(53)[7] For every subset A of Y holds $\overset{A}{\hookrightarrow} = \mathrm{id}_Y \restriction A$.

(54) For every subset A of Y holds $\mathrm{dom}(\overset{A}{\hookrightarrow}) = A$ and $\mathrm{rng}(\overset{A}{\hookrightarrow}) = A$.

(55) For every subset A of Y such that $x \in A$ holds $(\overset{A}{\hookrightarrow})(x) = x$.

(56) For every subset A of Y such that $x \in A$ holds $(\overset{A}{\hookrightarrow})(x) \in Y$.

Let us consider X, Y. The functor $\pi_1(X \times Y)$ yields a function and is defined as follows:

(Def. 5) $\mathrm{dom}\, \pi_1(X \times Y) = [:X, Y:]$ and for all x, y such that $x \in X$ and $y \in Y$ holds $\pi_1(X \times Y)(\langle x, y\rangle) = x$.

The functor $\pi_2(X \times Y)$ yields a function and is defined as follows:

(Def. 6) $\mathrm{dom}\, \pi_2(X \times Y) = [:X, Y:]$ and for all x, y such that $x \in X$ and $y \in Y$ holds $\pi_2(X \times Y)(\langle x, y\rangle) = y$.

[5] The propositions (40) and (41) have been removed.
[6] The propositions (44)–(46) have been removed.
[7] The propositions (50)–(52) have been removed.

We now state four propositions:

(59)[8] $\operatorname{rng}\pi_1(X \times Y) \subseteq X$.

(60) If $Y \neq \emptyset$, then $\operatorname{rng}\pi_1(X \times Y) = X$.

(61) $\operatorname{rng}\pi_2(X \times Y) \subseteq Y$.

(62) If $X \neq \emptyset$, then $\operatorname{rng}\pi_2(X \times Y) = Y$.

Let us consider X, Y. Then $\pi_1(X \times Y)$ is a function from $[:X, Y:]$ into X. Then $\pi_2(X \times Y)$ is a function from $[:X, Y:]$ into Y.

Let us consider X. The functor δ_X yielding a function is defined as follows:

(Def. 7) $\operatorname{dom}(\delta_X) = X$ and for every x such that $x \in X$ holds $\delta_X(x) = \langle x, x \rangle$.

The following proposition is true

(66)[9] $\operatorname{rng}(\delta_X) \subseteq [:X, X:]$.

Let us consider X. Then δ_X is a function from X into $[:X, X:]$.

Let f, g be functions. The functor $\langle f, g \rangle$ yielding a function is defined as follows:

(Def. 8) $\operatorname{dom}\langle f, g \rangle = \operatorname{dom} f \cap \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom}\langle f, g \rangle$ holds $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$.

One can prove the following propositions:

(68)[10] For all functions f, g such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$.

(69) For all functions f, g such that $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ and $x \in X$ holds $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$.

(70) For all functions f, g such that $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ holds $\operatorname{dom}\langle f, g \rangle = X$.

(71) For all functions f, g holds $\operatorname{rng}\langle f, g \rangle \subseteq [:\operatorname{rng} f, \operatorname{rng} g:]$.

(72) For all functions f, g such that $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{rng} f \subseteq Y$ and $\operatorname{rng} g \subseteq Z$ holds $\pi_1(Y \times Z) \cdot \langle f, g \rangle = f$ and $\pi_2(Y \times Z) \cdot \langle f, g \rangle = g$.

(73) $\langle \pi_1(X \times Y), \pi_2(X \times Y) \rangle = \mathrm{id}_{[:X, Y:]}$.

(74) For all functions f, g, h, k such that $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{dom} k = \operatorname{dom} h$ and $\langle f, g \rangle = \langle k, h \rangle$ holds $f = k$ and $g = h$.

(75) For all functions f, g, h holds $\langle f \cdot h, g \cdot h \rangle = \langle f, g \rangle \cdot h$.

(76) For all functions f, g holds $\langle f, g \rangle^{\circ} A \subseteq [:f^{\circ} A, g^{\circ} A:]$.

(77) For all functions f, g holds $\langle f, g \rangle^{-1}([:B, C:]) = f^{-1}(B) \cap g^{-1}(C)$.

(78) Let f be a function from X into Y and g be a function from X into Z. Suppose if $Y = \emptyset$, then $X = \emptyset$ and if $Z = \emptyset$, then $X = \emptyset$. Then $\langle f, g \rangle$ is a function from X into $[:Y, Z:]$.

Let us consider X, D_1, D_2, let f_1 be a function from X into D_1, and let f_2 be a function from X into D_2. Then $\langle f_1, f_2 \rangle$ is a function from X into $[:D_1, D_2:]$.

Next we state several propositions:

(79) Let f_1 be a function from C into D_1, f_2 be a function from C into D_2, and c be an element of C. Then $\langle f_1, f_2 \rangle(c) = \langle f_1(c), f_2(c) \rangle$.

[8] The propositions (57) and (58) have been removed.
[9] The propositions (63)–(65) have been removed.
[10] The proposition (67) has been removed.

(80) For every function f from X into Y and for every function g from X into Z holds $\mathrm{rng}\langle f,g\rangle \subseteq [:Y, Z:]$.

(81) Let f be a function from X into Y and g be a function from X into Z. Suppose if $Y = \emptyset$, then $X = \emptyset$ and if $Z = \emptyset$, then $X = \emptyset$. Then $\pi_1(Y \times Z) \cdot \langle f,g\rangle = f$ and $\pi_2(Y \times Z) \cdot \langle f,g\rangle = g$.

(82) For every function f from X into D_1 and for every function g from X into D_2 holds $\pi_1(D_1 \times D_2) \cdot \langle f,g\rangle = f$ and $\pi_2(D_1 \times D_2) \cdot \langle f,g\rangle = g$.

(83) Let f_1, f_2 be functions from X into Y and g_1, g_2 be functions from X into Z. Suppose if $Y = \emptyset$, then $X = \emptyset$ and if $Z = \emptyset$, then $X = \emptyset$ and $\langle f_1,g_1\rangle = \langle f_2,g_2\rangle$. Then $f_1 = f_2$ and $g_1 = g_2$.

(84) Let f_1, f_2 be functions from X into D_1 and g_1, g_2 be functions from X into D_2. If $\langle f_1,g_1\rangle = \langle f_2,g_2\rangle$, then $f_1 = f_2$ and $g_1 = g_2$.

Let f, g be functions. The functor $[:f, g:]$ yielding a function is defined by:

(Def. 9) $\mathrm{dom}[:f, g:] = [:\mathrm{dom}\, f, \mathrm{dom}\, g:]$ and for all x, y such that $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, g$ holds $[:f, g:](\langle x, y\rangle) = \langle f(x), g(y)\rangle$.

We now state a number of propositions:

(86)[11] For all functions f, g and for all x, y such that $\langle x, y\rangle \in [:\mathrm{dom}\, f, \mathrm{dom}\, g:]$ holds $[:f, g:](\langle x, y\rangle) = \langle f(x), g(y)\rangle$.

(87) For all functions f, g holds $[:f, g:] = \langle f \cdot \pi_1(\mathrm{dom}\, f \times \mathrm{dom}\, g), g \cdot \pi_2(\mathrm{dom}\, f \times \mathrm{dom}\, g)\rangle$.

(88) For all functions f, g holds $\mathrm{rng}[:f, g:] = [:\mathrm{rng}\, f, \mathrm{rng}\, g:]$.

(89) For all functions f, g such that $\mathrm{dom}\, f = X$ and $\mathrm{dom}\, g = X$ holds $\langle f,g\rangle = [:f, g:] \cdot \delta_X$.

(90) $[:\mathrm{id}_X, \mathrm{id}_Y:] = \mathrm{id}_{[:X,Y:]}$.

(91) For all functions f, g, h, k holds $[:f, h:] \cdot \langle g,k\rangle = \langle f \cdot g, h \cdot k\rangle$.

(92) For all functions f, g, h, k holds $[:f, h:] \cdot [:g, k:] = [:f \cdot g, h \cdot k:]$.

(93) For all functions f, g holds $[:f, g:]^\circ[:B, A:] = [:f^\circ B, g^\circ A:]$.

(94) For all functions f, g holds $[:f, g:]^{-1}([:B, A:]) = [:f^{-1}(B), g^{-1}(A):]$.

(95) Let f be a function from X into Y and g be a function from V into Z. Then $[:f, g:]$ is a function from $[:X, V:]$ into $[:Y, Z:]$.

Let us consider X_1, X_2, Y_1, Y_2, let f_1 be a function from X_1 into Y_1, and let f_2 be a function from X_2 into Y_2. Then $[:f_1, f_2:]$ is a function from $[:X_1, X_2:]$ into $[:Y_1, Y_2:]$.

Next we state several propositions:

(96) Let f_1 be a function from C_1 into D_1, f_2 be a function from C_2 into D_2, c_1 be an element of C_1, and c_2 be an element of C_2. Then $[:f_1, f_2:](\langle c_1, c_2\rangle) = \langle f_1(c_1), f_2(c_2)\rangle$.

(97) Let f_1 be a function from X_1 into Y_1 and f_2 be a function from X_2 into Y_2. If if $Y_1 = \emptyset$, then $X_1 = \emptyset$ and if $Y_2 = \emptyset$, then $X_2 = \emptyset$, then $[:f_1, f_2:] = \langle f_1 \cdot \pi_1(X_1 \times X_2), f_2 \cdot \pi_2(X_1 \times X_2)\rangle$.

(98) For every function f_1 from X_1 into D_1 and for every function f_2 from X_2 into D_2 holds $[:f_1, f_2:] = \langle f_1 \cdot \pi_1(X_1 \times X_2), f_2 \cdot \pi_2(X_1 \times X_2)\rangle$.

(99) Let f_1 be a function from X into Y_1 and f_2 be a function from X into Y_2. If if $Y_1 = \emptyset$, then $X = \emptyset$ and if $Y_2 = \emptyset$, then $X = \emptyset$, then $\langle f_1,f_2\rangle = [:f_1, f_2:] \cdot \delta_X$.

(100) For every function f_1 from X into D_1 and for every function f_2 from X into D_2 holds $\langle f_1,f_2\rangle = [:f_1, f_2:] \cdot \delta_X$.

[11] The proposition (85) has been removed.

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Received May 9, 1989

Published January 2, 2004