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The Sequential Closure Operator in Sequential and Frechet Spaces

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The articles [19], [23], [21], [2], [3], [15], [9], [24], [6], [7], [17], [8], [4], [1], [14], [13], [12], [5], [10], [16], [22], [11], [20], and [18] provide the notation and terminology for this paper.

1. THE PROPERTIES OF SEQUENCES AND SUBSEQUENCES

Let T be a non empty 1-sorted structure, let f be a function from $\mathbb{N}$ into $\mathbb{N}$, and let S be a sequence of T. Then $S \cdot f$ is a sequence of T.

Next we state two propositions:

(1) Let T be a non empty 1-sorted structure, S be a sequence of T, and N_1 be an increasing sequence of naturals. Then $S \cdot N_1$ is a sequence of T.

(2) For every sequence R_1 of real numbers such that $R_1 = \mathrm{id}_{\mathbb{N}}$ holds R_1 is an increasing sequence of naturals.

Let T be a non empty 1-sorted structure and let S be a sequence of T. A sequence of T is called a subsequence of S if:

(Def. 1) There exists an increasing sequence N_1 of naturals such that it $= S \cdot N_1$.

The following propositions are true:

(3) For every non empty 1-sorted structure T holds every sequence S of T is a subsequence of S.

(4) For every non empty 1-sorted structure T and for every sequence S of T and for every subsequence S_1 of S holds $\mathrm{rng}\, S_1 \subseteq \mathrm{rng}\, S$.

Let T be a non empty 1-sorted structure, let N_1 be an increasing sequence of naturals, and let S be a sequence of T. Then $S \cdot N_1$ is a subsequence of S.

One can prove the following proposition

(5) Let T be a non empty 1-sorted structure, S_1 be a sequence of T, and S_2 be a subsequence of S_1. Then every subsequence of S_2 is a subsequence of S_1.

In this article we present several logical schemes. The scheme *SubSeqChoice* deals with a non empty 1-sorted structure $\mathcal{A}$, a sequence $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a subsequence S_1 of $\mathcal{B}$ such that for every natural number n holds $\mathcal{P}[S_1(n)]$

provided the parameters satisfy the following condition:

- For every natural number n there exists a natural number m and there exists a point x of $\mathcal{A}$ such that $n \leq m$ and $x = \mathcal{B}(m)$ and $\mathcal{P}[x]$.

The scheme *SubSeqChoice1* deals with a non empty topological structure $\mathcal{A}$, a sequence $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a subsequence S_1 of $\mathcal{B}$ such that for every natural number n holds $\mathcal{P}[S_1(n)]$

provided the parameters meet the following requirement:

- For every natural number n there exists a natural number m and there exists a point x of $\mathcal{A}$ such that $n \leq m$ and $x = \mathcal{B}(m)$ and $\mathcal{P}[x]$.

We now state several propositions:

(6) Let T be a non empty 1-sorted structure, S be a sequence of T, and A be a subset of T. Suppose that for every subsequence S_1 of S holds $\operatorname{rng} S_1 \not\subseteq A$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $S(m) \notin A$.

(7) Let T be a non empty 1-sorted structure, S be a sequence of T, and A, B be subsets of T. If $\operatorname{rng} S \subseteq A \cup B$, then there exists a subsequence S_1 of S such that $\operatorname{rng} S_1 \subseteq A$ or $\operatorname{rng} S_1 \subseteq B$.

(8) Let T be a non empty topological space. Suppose that for every sequence S of T and for all points x_1, x_2 of T such that $x_1 \in \operatorname{Lim} S$ and $x_2 \in \operatorname{Lim} S$ holds $x_1 = x_2$. Then T is a T_1 space.

(9) Let T be a non empty topological space. Suppose T is a T_2 space. Let S be a sequence of T and x_1, x_2 be points of T. If $x_1 \in \operatorname{Lim} S$ and $x_2 \in \operatorname{Lim} S$, then $x_1 = x_2$.

(10) Let T be a non empty topological space. Suppose T is first-countable. Then T is a T_2 space if and only if for every sequence S of T and for all points x_1, x_2 of T such that $x_1 \in \operatorname{Lim} S$ and $x_2 \in \operatorname{Lim} S$ holds $x_1 = x_2$.

(11) For every non empty topological structure T and for every sequence S of T such that S is not convergent holds $\operatorname{Lim} S = \emptyset$.

(12) Let T be a non empty topological space and A be a subset of T. If A is closed, then for every sequence S of T such that $\operatorname{rng} S \subseteq A$ holds $\operatorname{Lim} S \subseteq A$.

(13) Let T be a non empty topological structure, S be a sequence of T, and x be a point of T. Suppose S is not convergent to x. Then there exists a subsequence S_1 of S such that for every subsequence S_2 of S_1 holds S_2 is not convergent to x.

2. The Continuous Maps

Next we state two propositions:

(14) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose f is continuous. Let S_1 be a sequence of T_1 and S_2 be a sequence of T_2. If $S_2 = f \cdot S_1$, then $f^{\circ} \operatorname{Lim} S_1 \subseteq \operatorname{Lim} S_2$.

(15) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose T_1 is sequential. Then f is continuous if and only if for every sequence S_1 of T_1 and for every sequence S_2 of T_2 such that $S_2 = f \cdot S_1$ holds $f^{\circ} \operatorname{Lim} S_1 \subseteq \operatorname{Lim} S_2$.

3. The Sequential Closure Operator

Let T be a non empty topological structure and let A be a subset of T. The functor $\operatorname{Cl}_{\mathrm{Seq}} A$ yields a subset of T and is defined by:

(Def. 2) For every point x of T holds $x \in \mathrm{Cl}_{\mathrm{Seq}} A$ iff there exists a sequence S of T such that $\mathrm{rng}\, S \subseteq A$ and $x \in \mathrm{Lim}\, S$.

One can prove the following propositions:

(16) Let T be a non empty topological structure, A be a subset of T, S be a sequence of T, and x be a point of T. If $\mathrm{rng}\, S \subseteq A$ and $x \in \mathrm{Lim}\, S$, then $x \in \overline{A}$.

(17) For every non empty topological structure T and for every subset A of T holds $\mathrm{Cl}_{\mathrm{Seq}} A \subseteq \overline{A}$.

(18) Let T be a non empty topological structure, S be a sequence of T, S_1 be a subsequence of S, and x be a point of T. If S is convergent to x, then S_1 is convergent to x.

(19) Let T be a non empty topological structure, S be a sequence of T, and S_1 be a subsequence of S. Then $\mathrm{Lim}\, S \subseteq \mathrm{Lim}\, S_1$.

(20) For every non empty topological structure T holds $\mathrm{Cl}_{\mathrm{Seq}}(\emptyset_T) = \emptyset$.

(21) For every non empty topological structure T and for every subset A of T holds $A \subseteq \mathrm{Cl}_{\mathrm{Seq}} A$.

(22) For every non empty topological structure T and for all subsets A, B of T holds $\mathrm{Cl}_{\mathrm{Seq}} A \cup \mathrm{Cl}_{\mathrm{Seq}} B = \mathrm{Cl}_{\mathrm{Seq}}(A \cup B)$.

(23) For every non empty topological structure T holds T is Frechet iff for every subset A of T holds $\overline{A} = \mathrm{Cl}_{\mathrm{Seq}} A$.

(24) Let T be a non empty topological space. Suppose T is Frechet. Let A, B be subsets of T. Then $\mathrm{Cl}_{\mathrm{Seq}}(\emptyset_T) = \emptyset$ and $A \subseteq \mathrm{Cl}_{\mathrm{Seq}} A$ and $\mathrm{Cl}_{\mathrm{Seq}}(A \cup B) = \mathrm{Cl}_{\mathrm{Seq}} A \cup \mathrm{Cl}_{\mathrm{Seq}} B$ and $\mathrm{Cl}_{\mathrm{Seq}} \mathrm{Cl}_{\mathrm{Seq}} A = \mathrm{Cl}_{\mathrm{Seq}} A$.

(25) Let T be a non empty topological space. Suppose T is sequential. If for every subset A of T holds $\mathrm{Cl}_{\mathrm{Seq}} \mathrm{Cl}_{\mathrm{Seq}} A = \mathrm{Cl}_{\mathrm{Seq}} A$, then T is Frechet.

(26) Let T be a non empty topological space. Suppose T is sequential. Then T is Frechet if and only if for all subsets A, B of T holds $\mathrm{Cl}_{\mathrm{Seq}}(\emptyset_T) = \emptyset$ and $A \subseteq \mathrm{Cl}_{\mathrm{Seq}} A$ and $\mathrm{Cl}_{\mathrm{Seq}}(A \cup B) = \mathrm{Cl}_{\mathrm{Seq}} A \cup \mathrm{Cl}_{\mathrm{Seq}} B$ and $\mathrm{Cl}_{\mathrm{Seq}} \mathrm{Cl}_{\mathrm{Seq}} A = \mathrm{Cl}_{\mathrm{Seq}} A$.

4. The Limit

Let T be a non empty topological space and let S be a sequence of T. Let us assume that there exists a point x of T such that $\mathrm{Lim}\, S = \{x\}$. The functor $\lim S$ yielding a point of T is defined by:

(Def. 3) S is convergent to $\lim S$.

The following propositions are true:

(27) Let T be a non empty topological space. Suppose T is a T_2 space. Let S be a sequence of T. If S is convergent, then there exists a point x of T such that $\mathrm{Lim}\, S = \{x\}$.

(28) Let T be a non empty topological space. Suppose T is a T_2 space. Let S be a sequence of T and x be a point of T. Then S is convergent to x if and only if S is convergent and $x = \lim S$.

(29) For every metric structure M holds every sequence of M is a sequence of M_{top}.

(30) For every non empty metric structure M holds every sequence of M_{top} is a sequence of M.

(31) Let M be a non empty metric space, S be a sequence of M, x be a point of M, S' be a sequence of M_{top}, and x' be a point of M_{top}. Suppose $S = S'$ and $x = x'$. Then S is convergent to x if and only if S' is convergent to x'.

(32) Let M be a non empty metric space, S_3 be a sequence of M, and S_4 be a sequence of M_{top}. If $S_3 = S_4$, then S_3 is convergent iff S_4 is convergent.

(33) Let M be a non empty metric space, S_3 be a sequence of M, and S_4 be a sequence of M_{top}. If $S_3 = S_4$ and S_3 is convergent, then $\lim S_3 = \lim S_4$.

5. The Cluster Points

Let T be a topological structure, let S be a sequence of T, and let x be a point of T. We say that x is a cluster point of S if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let O be a subset of T and n be a natural number. Suppose O is open and $x \in O$. Then there exists a natural number m such that $n \leq m$ and $S(m) \in O$.

One can prove the following propositions:

(34) Let T be a non empty topological structure, S be a sequence of T, and x be a point of T. If there exists a subsequence of S which is convergent to x, then x is a cluster point of S.

(35) Let T be a non empty topological structure, S be a sequence of T, and x be a point of T. If S is convergent to x, then x is a cluster point of S.

(36) Let T be a non empty topological structure, S be a sequence of T, x be a point of T, and Y be a subset of T. If $Y = \{y; y$ ranges over points of T: $x \in \overline{\{y\}}\}$ and $\mathrm{rng}\, S \subseteq Y$, then S is convergent to x.

(37) Let T be a non empty topological structure, S be a sequence of T, and x, y be points of T. Suppose that for every natural number n holds $S(n) = y$ and S is convergent to x. Then $x \in \overline{\{y\}}$.

(38) Let T be a non empty topological structure, x be a point of T, Y be a subset of T, and S be a sequence of T. Suppose $Y = \{y; y$ ranges over points of T: $x \in \overline{\{y\}}\}$ and $\mathrm{rng}\, S$ misses Y and S is convergent to x. Then there exists a subsequence of S which is one-to-one.

(39) Let T be a non empty topological structure and S_1, S_2 be sequences of T. Suppose $\mathrm{rng}\, S_2 \subseteq \mathrm{rng}\, S_1$ and S_2 is one-to-one. Then there exists a permutation P of $\mathbb{N}$ such that $S_2 \cdot P$ is a subsequence of S_1.

Now we present two schemes. The scheme *PermSeq* deals with a non empty 1-sorted structure $\mathcal{A}$, a sequence $\mathcal{B}$ of $\mathcal{A}$, a permutation $\mathcal{C}$ of $\mathbb{N}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a natural number n such that for every natural number m such that $n \leq m$ holds $\mathcal{P}[(\mathcal{B} \cdot \mathcal{C})(m)]$

provided the parameters meet the following requirement:

- There exists a natural number n such that for every natural number m and for every point x of $\mathcal{A}$ if $n \leq m$ and $x = \mathcal{B}(m)$, then $\mathcal{P}[x]$.

The scheme *PermSeq2* deals with a non empty topological structure $\mathcal{A}$, a sequence $\mathcal{B}$ of $\mathcal{A}$, a permutation $\mathcal{C}$ of $\mathbb{N}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a natural number n such that for every natural number m such that $n \leq m$ holds $\mathcal{P}[(\mathcal{B} \cdot \mathcal{C})(m)]$

provided the parameters satisfy the following condition:

- There exists a natural number n such that for every natural number m and for every point x of $\mathcal{A}$ if $n \leq m$ and $x = \mathcal{B}(m)$, then $\mathcal{P}[x]$.

One can prove the following propositions:

(40) Let T be a non empty topological structure, S be a sequence of T, P be a permutation of $\mathbb{N}$, and x be a point of T. If S is convergent to x, then $S \cdot P$ is convergent to x.

(41) Let n_0 be a natural number. Then there exists an increasing sequence N_1 of naturals such that for every natural number n holds $N_1(n) = n + n_0$.

(42) Let T be a non empty 1-sorted structure, S be a sequence of T, and n_0 be a natural number. Then there exists a subsequence S_1 of S such that for every natural number n holds $S_1(n) = S(n + n_0)$.

(43) Let T be a non empty topological structure, S be a sequence of T, x be a point of T, and S_1 be a subsequence of S. Suppose x is a cluster point of S and there exists a natural number n_0 such that for every natural number n holds $S_1(n) = S(n + n_0)$. Then x is a cluster point of S_1.

(44) Let T be a non empty topological structure, S be a sequence of T, and x be a point of T. If x is a cluster point of S, then $x \in \overline{\operatorname{rng} S}$.

(45) Let T be a non empty topological structure. Suppose T is Frechet. Let S be a sequence of T and x be a point of T. If x is a cluster point of S, then there exists a subsequence of S which is convergent to x.

6. Auxiliary Theorems

We now state several propositions:

(46) Let T be a non empty topological space. Suppose T is first-countable. Let x be a point of T. Then there exists a basis B of x and there exists a function S such that $\operatorname{dom} S = \mathbb{N}$ and $\operatorname{rng} S = B$ and for all natural numbers n, m such that $m \geq n$ holds $S(m) \subseteq S(n)$.

(47) For every non empty topological space T holds T is a T_1 space iff for every point p of T holds $\overline{\{p\}} = \{p\}$.

(48) For every non empty topological space T such that T is a T_2 space holds T is a T_1 space.

(49) Let T be a non empty topological space. Suppose T is not a T_1 space. Then there exist points x_1, x_2 of T and there exists a sequence S of T such that $S = \mathbb{N} \longmapsto x_1$ and $x_1 \neq x_2$ and S is convergent to x_2.

(50) For every function f such that $\operatorname{dom} f$ is infinite and f is one-to-one holds $\operatorname{rng} f$ is infinite.

(51) For every non empty finite subset X of $\mathbb{N}$ and for every natural number x such that $x \in X$ holds $x \leq \max X$.

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