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Segments of Natural Numbers and Finite Sequences[1]

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Summary. We define the notion of an initial segment of natural numbers and prove a number of their properties. Using this notion we introduce finite sequences, subsequences, the empty sequence, a sequence of a domain, and the operation of concatenation of two sequences.

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The articles [9], [7], [11], [4], [12], [6], [5], [3], [2], [10], [8], and [1] provide the notation and terminology for this paper.

1. Main Part

For simplicity, we adopt the following rules: k, l, m, n, k_1, k_2 denote natural numbers, a, b, c denote natural numbers, x, y, z, y_1, y_2, X denote sets, and f denotes a function.

Let n be a natural number. The functor $\mathrm{Seg}\,n$ yields a set and is defined as follows:

(Def. 1) $\mathrm{Seg}\,n = \{k : 1 \leq k \land k \leq n\}$.

Let n be a natural number. Then $\mathrm{Seg}\,n$ is a subset of $\mathbb{N}$.

Next we state several propositions:

(3)[1] $a \in \mathrm{Seg}\,b$ iff $1 \leq a$ and $a \leq b$.

(4) $\mathrm{Seg}\,0 = \emptyset$ and $\mathrm{Seg}\,1 = \{1\}$ and $\mathrm{Seg}\,2 = \{1,2\}$.

(5) $a = 0$ or $a \in \mathrm{Seg}\,a$.

(6) $a+1 \in \mathrm{Seg}(a+1)$.

(7) $a \leq b$ iff $\mathrm{Seg}\,a \subseteq \mathrm{Seg}\,b$.

(8) If $\mathrm{Seg}\,a = \mathrm{Seg}\,b$, then $a = b$.

(9) If $c \leq a$, then $\mathrm{Seg}\,c = \mathrm{Seg}\,c \cap \mathrm{Seg}\,a$ and $\mathrm{Seg}\,c = \mathrm{Seg}\,a \cap \mathrm{Seg}\,c$.

(10) If $\mathrm{Seg}\,c = \mathrm{Seg}\,c \cap \mathrm{Seg}\,a$ or $\mathrm{Seg}\,c = \mathrm{Seg}\,a \cap \mathrm{Seg}\,c$, then $c \leq a$.

(11) $\mathrm{Seg}\,a \cup \{a+1\} = \mathrm{Seg}(a+1)$.

Let I_1 be a binary relation. We say that I_1 is finite sequence-like if and only if:

[1]Supported by RPBP.III-24.C1.
[1] The propositions (1) and (2) have been removed.

(Def. 2) There exists n such that $\operatorname{dom} I_1 = \operatorname{Seg} n$.

One can check that there exists a function which is finite sequence-like.
A finite sequence is a finite sequence-like function.
In the sequel p, q, r denote finite sequences.
Let us consider n. Observe that $\operatorname{Seg} n$ is finite.
One can check that every function which is finite sequence-like is also finite.
Let us consider p. Then $\overline{\overline{p}}$ is a natural number and it can be characterized by the condition:

(Def. 3) $\operatorname{Seg} \overline{\overline{p}} = \operatorname{dom} p$.

We introduce $\operatorname{len} p$ as a synonym of $\overline{\overline{p}}$.
Let us consider p. Then $\operatorname{dom} p$ is a subset of $\mathbb{N}$.
One can prove the following two propositions:

(14)[2] $\emptyset$ is a finite sequence.

(15) If there exists k such that $\operatorname{dom} f \subseteq \operatorname{Seg} k$, then there exists p such that $f \subseteq p$.

In this article we present several logical schemes. The scheme *SeqEx* deals with a natural number $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists p such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{A}$ and for every k such that $k \in \operatorname{Seg} \mathcal{A}$ holds $\mathcal{P}[k, p(k)]$

provided the parameters satisfy the following conditions:

- For all k, y_1, y_2 such that $k \in \operatorname{Seg} \mathcal{A}$ and $\mathcal{P}[k, y_1]$ and $\mathcal{P}[k, y_2]$ holds $y_1 = y_2$, and
- For every k such that $k \in \operatorname{Seg} \mathcal{A}$ there exists x such that $\mathcal{P}[k, x]$.

The scheme *SeqLambda* deals with a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a finite sequence p such that $\operatorname{len} p = \mathcal{A}$ and for every k such that $k \in \operatorname{Seg} \mathcal{A}$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

One can prove the following propositions:

(16) If $z \in p$, then there exists k such that $k \in \operatorname{dom} p$ and $z = \langle k, p(k) \rangle$.

(17) If $\operatorname{dom} p = \operatorname{dom} q$ and for every k such that $k \in \operatorname{dom} p$ holds $p(k) = q(k)$, then $p = q$.

(18) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

(19) $p \restriction \operatorname{Seg} a$ is a finite sequence.

(20) If $\operatorname{rng} p \subseteq \operatorname{dom} f$, then $f \cdot p$ is a finite sequence.

(21) If $a \leq \operatorname{len} p$ and $q = p \restriction \operatorname{Seg} a$, then $\operatorname{len} q = a$ and $\operatorname{dom} q = \operatorname{Seg} a$.

Let D be a set. A finite sequence is called a finite sequence of elements of D if:

(Def. 4) $\operatorname{rng} \mathrm{it} \subseteq D$.

Let us observe that $\emptyset$ is finite sequence-like.
Let D be a set. Observe that there exists a partial function from $\mathbb{N}$ to D which is finite sequence-like.
Let D be a set. We see that the finite sequence of elements of D is a finite sequence-like partial function from $\mathbb{N}$ to D.
In the sequel D is a set.
Next we state two propositions:

(23)[3] For every finite sequence p of elements of D holds $p \restriction \operatorname{Seg} a$ is a finite sequence of elements of D.

[2] The propositions (12) and (13) have been removed.
[3] The proposition (22) has been removed.

(24) For every non empty set D there exists a finite sequence p of elements of D such that $\operatorname{len} p = a$.

Let us observe that there exists a finite sequence which is empty.

One can prove the following propositions:

(25) $\operatorname{len} p = 0$ iff $p = \emptyset$.

(26) $p = \emptyset$ iff $\operatorname{dom} p = \emptyset$.

(27) $p = \emptyset$ iff $\operatorname{rng} p = \emptyset$.

(29)[4] For every set D holds $\emptyset$ is a finite sequence of elements of D.

Let D be a set. One can check that there exists a finite sequence of elements of D which is empty.

Let us consider x. The functor $\langle x \rangle$ yields a set and is defined by:

(Def. 5) $\langle x \rangle = \{\langle 1, x \rangle\}$.

Let D be a set. The functor ε_D yields an empty finite sequence of elements of D and is defined as follows:

(Def. 6) $\varepsilon_D = \emptyset$.

The following proposition is true

(32)[5] $p = \varepsilon_D$ iff $\operatorname{len} p = 0$.

Let us consider p, q. The functor $p \frown q$ yields a finite sequence and is defined as follows:

(Def. 7) $\operatorname{dom}(p \frown q) = \operatorname{Seg}(\operatorname{len} p + \operatorname{len} q)$ and for every k such that $k \in \operatorname{dom} p$ holds $(p \frown q)(k) = p(k)$ and for every k such that $k \in \operatorname{dom} q$ holds $(p \frown q)(\operatorname{len} p + k) = q(k)$.

One can prove the following propositions:

(35)[6] $\operatorname{len}(p \frown q) = \operatorname{len} p + \operatorname{len} q$.

(36) If $\operatorname{len} p + 1 \leq k$ and $k \leq \operatorname{len} p + \operatorname{len} q$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(37) If $\operatorname{len} p < k$ and $k \leq \operatorname{len}(p \frown q)$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(38) If $k \in \operatorname{dom}(p \frown q)$, then $k \in \operatorname{dom} p$ or there exists n such that $n \in \operatorname{dom} q$ and $k = \operatorname{len} p + n$.

(39) $\operatorname{dom} p \subseteq \operatorname{dom}(p \frown q)$.

(40) If $x \in \operatorname{dom} q$, then there exists k such that $k = x$ and $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(41) If $k \in \operatorname{dom} q$, then $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(42) $\operatorname{rng} p \subseteq \operatorname{rng}(p \frown q)$.

(43) $\operatorname{rng} q \subseteq \operatorname{rng}(p \frown q)$.

(44) $\operatorname{rng}(p \frown q) = \operatorname{rng} p \cup \operatorname{rng} q$.

(45) $(p \frown q) \frown r = p \frown (q \frown r)$.

(46) If $p \frown r = q \frown r$ or $r \frown p = r \frown q$, then $p = q$.

(47) $p \frown \emptyset = p$ and $\emptyset \frown p = p$.

[4] The proposition (28) has been removed.

[5] The propositions (30) and (31) have been removed.

[6] The propositions (33) and (34) have been removed.

(48) If $p \frown q = \emptyset$, then $p = \emptyset$ and $q = \emptyset$.

Let D be a set and let p, q be finite sequences of elements of D. Then $p \frown q$ is a finite sequence of elements of D.

Let us consider x. Then $\langle x \rangle$ is a function and it can be characterized by the condition:

(Def. 8) $\operatorname{dom}\langle x \rangle = \operatorname{Seg} 1$ and $\langle x \rangle(1) = x$.

Let us consider x. Observe that $\langle x \rangle$ is function-like and relation-like.

Let us consider x. Observe that $\langle x \rangle$ is finite sequence-like.

We now state the proposition

(50)[7] Suppose $p \frown q$ is a finite sequence of elements of D. Then p is a finite sequence of elements of D and q is a finite sequence of elements of D.

Let us consider x, y. The functor $\langle x, y \rangle$ yields a set and is defined as follows:

(Def. 9) $\langle x, y \rangle = \langle x \rangle \frown \langle y \rangle$.

Let us consider z. The functor $\langle x, y, z \rangle$ yields a set and is defined as follows:

(Def. 10) $\langle x, y, z \rangle = \langle x \rangle \frown \langle y \rangle \frown \langle z \rangle$.

Let us consider x, y. Observe that $\langle x, y \rangle$ is function-like and relation-like. Let us consider z. One can check that $\langle x, y, z \rangle$ is function-like and relation-like.

Let us consider x, y. One can verify that $\langle x, y \rangle$ is finite sequence-like. Let us consider z. Observe that $\langle x, y, z \rangle$ is finite sequence-like.

We now state a number of propositions:

(52)[8] $\langle x \rangle = \{\langle 1, x \rangle\}$.

(55)[9] $p = \langle x \rangle$ iff $\operatorname{dom} p = \operatorname{Seg} 1$ and $\operatorname{rng} p = \{x\}$.

(56) $p = \langle x \rangle$ iff $\operatorname{len} p = 1$ and $\operatorname{rng} p = \{x\}$.

(57) $p = \langle x \rangle$ iff $\operatorname{len} p = 1$ and $p(1) = x$.

(58) $(\langle x \rangle \frown p)(1) = x$.

(59) $(p \frown \langle x \rangle)(\operatorname{len} p + 1) = x$.

(60) $\langle x, y, z \rangle = \langle x \rangle \frown \langle y, z \rangle$ and $\langle x, y, z \rangle = \langle x, y \rangle \frown \langle z \rangle$.

(61) $p = \langle x, y \rangle$ iff $\operatorname{len} p = 2$ and $p(1) = x$ and $p(2) = y$.

(62) $p = \langle x, y, z \rangle$ iff $\operatorname{len} p = 3$ and $p(1) = x$ and $p(2) = y$ and $p(3) = z$.

(63) If $p \neq \emptyset$, then there exist q, x such that $p = q \frown \langle x \rangle$.

Let D be a non empty set and let x be an element of D. Then $\langle x \rangle$ is a finite sequence of elements of D.

The scheme *IndSeq* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\emptyset]$, and
- For all p, x such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$.

We now state the proposition

(64) For all finite sequences p, q, r, s such that $p \frown q = r \frown s$ and $\operatorname{len} p \leq \operatorname{len} r$ there exists a finite sequence t such that $p \frown t = r$.

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[7] The proposition (49) has been removed.

[8] The proposition (51) has been removed.

[9] The propositions (53) and (54) have been removed.

Let D be a set. The functor D^* yields a set and is defined by:

(Def. 11) $x \in D^*$ iff x is a finite sequence of elements of D.

Let D be a set. One can verify that D^* is non empty.
The following proposition is true

(66)[10] $\emptyset \in D^*$.

The scheme *SepSeq* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:
There exists X such that for every x holds $x \in X$ iff there exists p such that $p \in \mathcal{A}^*$ and $\mathcal{P}[p]$ and $x = p$
for all values of the parameters.

Let I_1 be a function. We say that I_1 is finite subsequence-like if and only if:

(Def. 12) There exists k such that $\operatorname{dom} I_1 \subseteq \operatorname{Seg} k$.

One can check that there exists a function which is finite subsequence-like.
A finite subsequence is a finite subsequence-like function.
We now state two propositions:

(68)[11] Every finite sequence is a finite subsequence.

(69) $p \restriction X$ is a finite subsequence and $X \restriction p$ is a finite subsequence.

In the sequel p' is a finite subsequence.

Let us consider X. Let us assume that there exists k such that $X \subseteq \operatorname{Seg} k$. The functor $\operatorname{Sgm} X$ yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 13) $\operatorname{rng} \operatorname{Sgm} X = X$ and for all l, m, k_1, k_2 such that $1 \leq l$ and $l < m$ and $m \leq \operatorname{len} \operatorname{Sgm} X$ and $k_1 = (\operatorname{Sgm} X)(l)$ and $k_2 = (\operatorname{Sgm} X)(m)$ holds $k_1 < k_2$.

The following proposition is true

(71)[12] $\operatorname{rng} \operatorname{Sgm} \operatorname{dom} p' = \operatorname{dom} p'$.

Let us consider p'. The functor $\operatorname{Seq} p'$ yields a function and is defined as follows:

(Def. 14) $\operatorname{Seq} p' = p' \cdot \operatorname{Sgm} \operatorname{dom} p'$.

Let us consider p'. Note that $\operatorname{Seq} p'$ is finite sequence-like.
Next we state the proposition

(72) For every X such that there exists k such that $X \subseteq \operatorname{Seg} k$ holds $\operatorname{Sgm} X = \emptyset$ iff $X = \emptyset$.

2. Moved from [8], 1998

One can prove the following proposition

(73) D is finite iff there exists p such that $D = \operatorname{rng} p$.

One can verify that there exists a function which is finite and empty.
Let us note that there exists a function which is finite and non empty.
Let R be a finite binary relation. Observe that $\operatorname{rng} R$ is finite.

[10] The proposition (65) has been removed.
[11] The proposition (67) has been removed.
[12] The proposition (70) has been removed.

3. MOVED FROM [1], 1999

One can prove the following propositions:

(74) If $\mathrm{Seg}\,n \approx \mathrm{Seg}\,m$, then $n = m$.

(75) $\mathrm{Seg}\,n \approx n$.

(76) $\overline{\overline{\mathrm{Seg}\,n}} = \overline{\overline{n}}$.

(77) If X is finite, then there exists n such that $X \approx \mathrm{Seg}\,n$.

(78) For every natural number n holds $\mathrm{card}\,\mathrm{Seg}\,n = n$ and $\mathrm{card}\,n = n$ and $\mathrm{card}\,\overline{\overline{n}} = n$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord2.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[11] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[12] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

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