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Real Function One-Side Differentiability

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Summary. We define real function one-side differentiability and one-side continuity. Main properties of one-side differentiability function are proved. Connections between one-side differential and differential real function at the point are demonstrated.

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The articles [11], [1], [12], [2], [14], [5], [3], [4], [13], [7], [8], [10], [9], and [6] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: h, h_1, h_2 denote convergent to 0 sequences of real numbers, c denotes a constant sequence of real numbers, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, x_0, r, r_1, g, g_1, g_2 denote real numbers, n denotes a natural number, and a denotes a sequence of real numbers.

One can prove the following propositions:

(1) If there exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$.

(2) If there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$.

(3) Suppose for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1}\ (f \cdot (h + c) - f \cdot c)$ is convergent and $\{x_0\} \subseteq \operatorname{dom} f$. Let given h_1, h_2, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h_1 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_1(n) < 0$ and $\operatorname{rng}(h_2 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_2(n) < 0$. Then $\lim({h_1}^{-1}\ (f \cdot (h_1 + c) - f \cdot c)) = \lim({h_2}^{-1}\ (f \cdot (h_2 + c) - f \cdot c))$.

(4) Suppose for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1}\ (f \cdot (h + c) - f \cdot c)$ is convergent and $\{x_0\} \subseteq \operatorname{dom} f$. Let given h_1, h_2, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h_1 + c) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(h_2 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_1(n) > 0$ and for every n holds $h_2(n) > 0$. Then $\lim({h_1}^{-1}\ (f \cdot (h_1 + c) - f \cdot c)) = \lim({h_2}^{-1}\ (f \cdot (h_2 + c) - f \cdot c))$.

Let us consider f, x_0. We say that f is left continuous in x_0 if and only if:

(Def. 1) $x_0 \in \operatorname{dom} f$ and for every a such that $\operatorname{rng} a \subseteq\ ]-\infty, x_0[\ \cap \operatorname{dom} f$ and a is convergent and $\lim a = x_0$ holds $f \cdot a$ is convergent and $f(x_0) = \lim(f \cdot a)$.

We say that f is right continuous in x_0 if and only if:

(Def. 2) $x_0 \in \operatorname{dom} f$ and for every a such that $\operatorname{rng} a \subseteq\ ]x_0, +\infty[\ \cap \operatorname{dom} f$ and a is convergent and $\lim a = x_0$ holds $f \cdot a$ is convergent and $f(x_0) = \lim(f \cdot a)$.

We say that f is right differentiable in x_0 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) There exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \mathrm{dom} f$, and

(ii) for all h, c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1}\ (f \cdot (h + c) - f \cdot c)$ is convergent.

We say that f is left differentiable in x_0 if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) There exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \mathrm{dom} f$, and

(ii) for all h, c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1}\ (f \cdot (h + c) - f \cdot c)$ is convergent.

We now state four propositions:

(5) If f is left differentiable in x_0, then f is left continuous in x_0.

(6) Suppose f is left continuous in x_0 and $f(x_0) \neq g_2$ and there exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \mathrm{dom} f$. Then there exists r_1 such that $r_1 > 0$ and $[x_0 - r_1, x_0] \subseteq \mathrm{dom} f$ and for every g such that $g \in [x_0 - r_1, x_0]$ holds $f(g) \neq g_2$.

(7) If f is right differentiable in x_0, then f is right continuous in x_0.

(8) Suppose f is right continuous in x_0 and $f(x_0) \neq g_2$ and there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \mathrm{dom} f$. Then there exists r_1 such that $r_1 > 0$ and $[x_0, x_0 + r_1] \subseteq \mathrm{dom} f$ and for every g such that $g \in [x_0, x_0 + r_1]$ holds $f(g) \neq g_2$.

Let us consider x_0, f. Let us assume that f is left differentiable in x_0. The functor $f'_{-}(x_0)$ yields a real number and is defined by:

(Def. 5) For all h, c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom} f$ and for every n holds $h(n) < 0$ holds $f'_{-}(x_0) = \lim(h^{-1}\ (f \cdot (h + c) - f \cdot c))$.

Let us consider x_0, f. Let us assume that f is right differentiable in x_0. The functor $f'_{+}(x_0)$ yielding a real number is defined by:

(Def. 6) For all h, c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom} f$ and for every n holds $h(n) > 0$ holds $f'_{+}(x_0) = \lim(h^{-1}\ (f \cdot (h + c) - f \cdot c))$.

The following propositions are true:

(9) f is left differentiable in x_0 and $f'_{-}(x_0) = g$ if and only if the following conditions are satisfied:

(i) there exists r such that $0 < r$ and $[x_0 - r, x_0] \subseteq \mathrm{dom} f$, and

(ii) for all h, c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1}\ (f \cdot (h + c) - f \cdot c)$ is convergent and $\lim(h^{-1}\ (f \cdot (h + c) - f \cdot c)) = g$.

(10) Suppose f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0. Then $f_1 + f_2$ is left differentiable in x_0 and $(f_1 + f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) + {f_2}'_{-}(x_0)$.

(11) Suppose f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0. Then $f_1 - f_2$ is left differentiable in x_0 and $(f_1 - f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) - {f_2}'_{-}(x_0)$.

(12) Suppose f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0. Then $f_1\ f_2$ is left differentiable in x_0 and $(f_1\ f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) \cdot f_2(x_0) + {f_2}'_{-}(x_0) \cdot f_1(x_0)$.

(13) Suppose f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0 and $f_2(x_0) \neq 0$. Then $\frac{f_1}{f_2}$ is left differentiable in x_0 and $(\frac{f_1}{f_2})'_{-}(x_0) = \frac{{f_1}'_{-}(x_0) \cdot f_2(x_0) - {f_2}'_{-}(x_0) \cdot f_1(x_0)}{f_2(x_0)^{\mathbf{2}}}$.

(14) If f is left differentiable in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is left differentiable in x_0 and $(\frac{1}{f})'_{-}(x_0) = -\frac{f'_{-}(x_0)}{f(x_0)^{\mathbf{2}}}$.

(15) f is right differentiable in x_0 and $f'_+(x_0) = g_1$ if and only if the following conditions are satisfied:

(i) there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$, and

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1} (f \cdot (h + c) - f \cdot c)$ is convergent and $\lim(h^{-1} (f \cdot (h + c) - f \cdot c)) = g_1$.

(16) Suppose f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0. Then $f_1 + f_2$ is right differentiable in x_0 and $(f_1 + f_2)'_+(x_0) = {f_1}'_+(x_0) + {f_2}'_+(x_0)$.

(17) Suppose f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0. Then $f_1 - f_2$ is right differentiable in x_0 and $(f_1 - f_2)'_+(x_0) = {f_1}'_+(x_0) - {f_2}'_+(x_0)$.

(18) Suppose f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0. Then $f_1\ f_2$ is right differentiable in x_0 and $(f_1\ f_2)'_+(x_0) = {f_1}'_+(x_0) \cdot f_2(x_0) + {f_2}'_+(x_0) \cdot f_1(x_0)$.

(19) Suppose f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0 and $f_2(x_0) \neq 0$. Then $\frac{f_1}{f_2}$ is right differentiable in x_0 and $(\frac{f_1}{f_2})'_+(x_0) = \frac{{f_1}'_+(x_0) \cdot f_2(x_0) - {f_2}'_+(x_0) \cdot f_1(x_0)}{f_2(x_0)^2}$.

(20) If f is right differentiable in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is right differentiable in x_0 and $(\frac{1}{f})'_+(x_0) = -\frac{f'_+(x_0)}{f(x_0)^2}$.

(21) Suppose f is right differentiable in x_0 and left differentiable in x_0 and $f'_+(x_0) = f'_-(x_0)$. Then f is differentiable in x_0 and $f'(x_0) = f'_+(x_0)$ and $f'(x_0) = f'_-(x_0)$.

(22) Suppose f is differentiable in x_0. Then f is right differentiable in x_0 and left differentiable in x_0 and $f'(x_0) = f'_+(x_0)$ and $f'(x_0) = f'_-(x_0)$.

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