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Fundamental Types of Metric Affine Spaces

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Summary. We distinguish in the class of metric affine spaces some fundamental types of them. First we can assume the underlying affine space to satisfy classical affine configurational axiom; thus we come to Pappian, Desarguesian, Moufangian, and translation spaces. Next we distinguish the spaces satisfying theorem on three perpendiculars and the homogeneous spaces; these properties directly refer to some axioms involving orthogonality. Some known relationships between the introduced classes of structures are established. We also show that the commonly investigated models of metric affine geometry constructed in a real linear space with the help of a symmetric bilinear form belong to all the classes introduced in the paper.

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The articles [8], [1], [7], [3], [4], [2], [6], and [5] provide the notation and terminology for this paper.

Let I_1 be a metric affine space. We say that I_1 is Euclidean if and only if:

(Def. 1) For all elements a, b, c, d of the carrier of I_1 such that $a,b \perp c,d$ and $b,c \perp a,d$ holds $b,d \perp a,c$.

Let I_1 be a metric affine space. We say that I_1 is Pappian if and only if:

(Def. 2) The affine reduct of I_1 is Pappian.

Let I_1 be a metric affine space. We say that I_1 is Desarguesian if and only if:

(Def. 3) The affine reduct of I_1 is Desarguesian.

Let I_1 be a metric affine space. We say that I_1 is Fanoian if and only if:

(Def. 4) The affine reduct of I_1 is Fanoian.

Let I_1 be a metric affine space. We say that I_1 is Moufangian if and only if:

(Def. 5) The affine reduct of I_1 is Moufangian.

Let I_1 be a metric affine space. We say that I_1 is translation if and only if:

(Def. 6) The affine reduct of I_1 is translational.

Let I_1 be a metric affine space. We say that I_1 is homogeneous if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let o, a, a_1, b, b_1, c, c_1 be elements of I_1. Suppose $o,a \perp o,a_1$ and $o,b \perp o,b_1$ and $o,c \perp o,c_1$ and $a,b \perp a_1,b_1$ and $a,c \perp a_1,c_1$ and $o,c \nparallel o,a$ and $o,a \nparallel o,b$. Then $b,c \perp b_1,c_1$.

In the sequel M_1 denotes a metric affine plane and M_2 denotes a metric affine space.
We now state a number of propositions:

(1) For all elements a, b, c of M_2 such that not $\mathbf{L}(a,b,c)$ holds $a \neq b$ and $b \neq c$ and $a \neq c$.

(2) Let a, b, c, d be elements of M_1 and K be a subset of the carrier of M_1. If $a,b \perp K$ and $c,d \perp K$, then $a,b \upuparrows c,d$ and $a,b \upuparrows d,c$.

(3) Let a, b be elements of M_1 and A, K be subsets of the carrier of M_1. If $a \neq b$ and if $a,b \perp K$ or $b,a \perp K$ and if $a,b \perp A$ or $b,a \perp A$, then $K \parallel A$.

(4) For all elements x, y, z of M_2 such that $\mathbf{L}(x,y,z)$ holds $\mathbf{L}(x,z,y)$ and $\mathbf{L}(y,x,z)$ and $\mathbf{L}(y,z,x)$ and $\mathbf{L}(z,x,y)$ and $\mathbf{L}(z,y,x)$.

(5) For all elements a, b, c of M_1 such that not $\mathbf{L}(a,b,c)$ there exists an element d of M_1 such that $d,a \perp b,c$ and $d,b \perp a,c$.

(6) For all elements a, b, c, d_1, d_2 of M_1 such that not $\mathbf{L}(a,b,c)$ and $d_1,a \perp b,c$ and $d_1,b \perp a,c$ and $d_2,a \perp b,c$ and $d_2,b \perp a,c$ holds $d_1 = d_2$.

(7) For all elements a, b, c, d of M_1 such that $a,b \perp c,d$ and $b,c \perp a,d$ and $\mathbf{L}(a,b,c)$ holds $a = c$ or $a = b$ or $b = c$.

(8) M_1 is Euclidean iff theorem on three perpendiculars holds in M_1.

(9) M_1 is homogeneous iff orthogonal version of Desargues Axiom holds in M_1.

(10) M_1 is Pappian iff Pappos Axiom holds in M_1.

(11) M_1 is Desarguesian iff Desargues Axiom holds in M_1.

(12) M_1 is Moufangian iff trapezium variant of Desargues Axiom holds in M_1.

(13) M_1 is translation iff minor Desargues Axiom holds in M_1.

(14) If M_1 is homogeneous, then M_1 is Desarguesian.

(15) If M_1 is Euclidean and Desarguesian, then M_1 is Pappian.

We adopt the following convention: V is a real linear space and w, y, u, v are vectors of V.
One can prove the following propositions:

(16) Let o, c, c_1, a, a_1, a_2 be elements of M_1. Suppose not $\mathbf{L}(o,c,a)$ and $o \neq c_1$ and $o,c \perp o,c_1$ and $o,a \perp o,a_1$ and $o,a \perp o,a_2$ and $c,a \perp c_1,a_1$ and $c,a \perp c_1,a_2$. Then $a_1 = a_2$.

(17) For all elements o, c, c_1, a of M_1 such that not $\mathbf{L}(o,c,a)$ and $o \neq c_1$ there exists an element a_1 of M_1 such that $o,a \perp o,a_1$ and $c,a \perp c_1,a_1$.

(18) Let a, b be real numbers. Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y and $u = a \cdot w + b \cdot y$. Then there exists a real number c such that $c \neq 0$ and $v = c \cdot b \cdot w + (-c \cdot a) \cdot y$.

(19) Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y. Then there exists a real number c such that for all real numbers a, b holds

(i) $a \cdot w + b \cdot y$, $c \cdot b \cdot w + (-c \cdot a) \cdot y$ are orthogonal w.r.t. w, y, and

(ii) $(a \cdot w + b \cdot y) - u$, $(c \cdot b \cdot w + (-c \cdot a) \cdot y) - v$ are orthogonal w.r.t. w, y.

(21)[1] If w, y span the space and $M_1 = \mathbf{AMSp}(V,w,y)$, then LIN holds in M_1.

[1] The proposition (20) has been removed.

(22) Let o, a, a_1, b, b_1, c, c_1 be elements of M_1. Suppose $o,a \perp o,a_1$ and $o,b \perp o,b_1$ and $o,c \perp o,c_1$ and $a,b \perp a_1,b_1$ and $a,c \perp a_1,c_1$ and $o,c \nparallel o,a$ and $o,a \nparallel o,b$ and $o = a_1$. Then $b,c \perp b_1,c_1$.

(23) If w, y span the space and $M_1 = \mathbf{AMSp}(V,w,y)$, then M_1 is homogeneous.

Let us observe that there exists a metric affine plane which is Euclidean, Pappian, Desarguesian, homogeneous, translation, Fanoian, and Moufangian.

Let us observe that there exists a metric affine space which is Euclidean, Pappian, Desarguesian, homogeneous, translation, Fanoian, and Moufangian.

The following proposition is true

(24) Suppose w, y span the space and $M_1 = \mathbf{AMSp}(V,w,y)$. Then M_1 is an Euclidean Pappian Desarguesian homogeneous translation Fanoian Moufangian metric affine plane.

Let M_1 be a Pappian metric affine plane. Observe that the affine reduct of M_1 is Pappian.

Let M_1 be a Desarguesian metric affine plane. Note that the affine reduct of M_1 is Desarguesian.

Let M_1 be a Moufangian metric affine plane. Observe that the affine reduct of M_1 is Moufangian.

Let M_1 be a translation metric affine plane. Observe that the affine reduct of M_1 is translational.

Let M_1 be a Fanoian metric affine plane. Observe that the affine reduct of M_1 is Fanoian.

Let M_1 be a homogeneous metric affine plane. Note that the affine reduct of M_1 is Desarguesian.

Let M_1 be an Euclidean Desarguesian metric affine plane. One can check that the affine reduct of M_1 is Pappian.

Let M_1 be a Pappian metric affine space. Observe that the affine reduct of M_1 is Pappian.

Let M_1 be a Desarguesian metric affine space. One can verify that the affine reduct of M_1 is Desarguesian.

Let M_1 be a Moufangian metric affine space. Note that the affine reduct of M_1 is Moufangian.

Let M_1 be a translation metric affine space. Note that the affine reduct of M_1 is translational.

Let M_1 be a Fanoian metric affine space. Note that the affine reduct of M_1 is Fanoian.

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