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Lines in n-Dimensional Euclidean Spaces

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Summary. In this paper, we define the line of n-dimensional Euclidian space and we introduce basic properties of affine space on this space. Next, we define the inner product of elements of this space. At the end, we introduce orthogonality of lines of this space.

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The articles [12], [5], [15], [2], [13], [8], [4], [11], [10], [3], [6], [1], [14], [7], and [9] provide the notation and terminology for this paper.

We follow the rules: a, b, l_1 denote real numbers, n denotes a natural number, and x, x_1, x_2, y_1, y_2 denote elements of $\mathcal{R}^n$.

The following propositions are true:

(1) $0 \cdot x + x = x$ and $x + \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(2) $a \cdot \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(3) $1 \cdot x = x$ and $0 \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(4) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

(5) If $a \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, then $a = 0$ or $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(6) $a \cdot (x_1 + x_2) = a \cdot x_1 + a \cdot x_2$.

(7) $(a + b) \cdot x = a \cdot x + b \cdot x$.

(8) If $a \cdot x_1 = a \cdot x_2$, then $a = 0$ or $x_1 = x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $\mathrm{Line}(x_1, x_2)$ yielding a subset of $\mathcal{R}^n$ is defined by:

(Def. 1) $\mathrm{Line}(x_1, x_2) = \{(1 - l_1) \cdot x_1 + l_1 \cdot x_2\}$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. One can verify that $\mathrm{Line}(x_1, x_2)$ is non empty.

Next we state the proposition

(9) $\mathrm{Line}(x_1, x_2) = \mathrm{Line}(x_2, x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us observe that the functor $\mathrm{Line}(x_1,x_2)$ is commutative.

Next we state three propositions:

(10) $x_1 \in \mathrm{Line}(x_1,x_2)$ and $x_2 \in \mathrm{Line}(x_1,x_2)$.

(11) If $y_1 \in \mathrm{Line}(x_1,x_2)$ and $y_2 \in \mathrm{Line}(x_1,x_2)$, then $\mathrm{Line}(y_1,y_2) \subseteq \mathrm{Line}(x_1,x_2)$.

(12) If $y_1 \in \mathrm{Line}(x_1,x_2)$ and $y_2 \in \mathrm{Line}(x_1,x_2)$ and $y_1 \neq y_2$, then $\mathrm{Line}(x_1,x_2) \subseteq \mathrm{Line}(y_1,y_2)$.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is line if and only if:

(Def. 2) There exist x_1, x_2 such that $x_1 \neq x_2$ and $A = \mathrm{Line}(x_1,x_2)$.

We introduce A is a line as a synonym of A is line.

Next we state three propositions:

(13) Let A, C be subsets of $\mathcal{R}^n$ and given x_1, x_2. Suppose A is a line and C is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \in C$ and $x_2 \in C$. Then $x_1 = x_2$ or $A = C$.

(14) For every subset A of $\mathcal{R}^n$ such that A is a line there exist x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$.

(15) For every subset A of $\mathcal{R}^n$ such that A is a line there exists x_2 such that $x_1 \neq x_2$ and $x_2 \in A$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{Rn2Fin}(x)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 3) $\mathrm{Rn2Fin}(x) = x$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $|x|$ yielding a real number is defined as follows:

(Def. 4) $|x| = |\mathrm{Rn2Fin}(x)|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $|(x_1,x_2)|$ yields a real number and is defined by:

(Def. 5) $|(x_1,x_2)| = |(\mathrm{Rn2Fin}(x_1),\mathrm{Rn2Fin}(x_2))|$.

Let us observe that the functor $|(x_1,x_2)|$ is commutative.

One can prove the following propositions:

(16) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1,x_2)| = \frac{1}{4} \cdot (|x_1+x_2|^{\mathbf{2}} - |x_1-x_2|^{\mathbf{2}})$.

(17) For every element x of $\mathcal{R}^n$ holds $|(x,x)| \geq 0$.

(18) For every element x of $\mathcal{R}^n$ holds $|x|^{\mathbf{2}} = |(x,x)|$.

(19) For every element x of $\mathcal{R}^n$ holds $0 \leq |x|$.

(20) For every element x of $\mathcal{R}^n$ holds $|x| = \sqrt{|(x,x)|}$.

(21) For every element x of $\mathcal{R}^n$ holds $|(x,x)| = 0$ iff $|x| = 0$.

(22) For every element x of $\mathcal{R}^n$ holds $|(x,x)| = 0$ iff $x = \langle \underbrace{0,\ldots,0}_{n} \rangle$.

(23) For every element x of $\mathcal{R}^n$ holds $|(x,\langle \underbrace{0,\ldots,0}_{n} \rangle)| = 0$.

(24) For every element x of $\mathcal{R}^n$ holds $|(\langle \underbrace{0,\ldots,0}_{n} \rangle,x)| = 0$.

(25) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1+x_2,x_3)| = |(x_1,x_3)| + |(x_2,x_3)|$.

(26) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(a\cdot x_1,x_2)| = a\cdot|(x_1,x_2)|$.

(27) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(x_1,a\cdot x_2)| = a\cdot|(x_1,x_2)|$.

(28) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1,x_2)| = -|(x_1,x_2)|$.

(29) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1,-x_2)| = -|(x_1,x_2)|$.

(30) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1,-x_2)| = |(x_1,x_2)|$.

(31) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1-x_2,x_3)| = |(x_1,x_3)| - |(x_2,x_3)|$.

(32) For all real numbers a, b and for all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(a\cdot x_1+b\cdot x_2,x_3)| = a\cdot|(x_1,x_3)| + b\cdot|(x_2,x_3)|$.

(33) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1,y_1+y_2)| = |(x_1,y_1)| + |(x_1,y_2)|$.

(34) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1,y_1-y_2)| = |(x_1,y_1)| - |(x_1,y_2)|$.

(35) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1+x_2,y_1+y_2)| = |(x_1,y_1)| + |(x_1,y_2)| + |(x_2,y_1)| + |(x_2,y_2)|$.

(36) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1-x_2,y_1-y_2)| = (|(x_1,y_1)| - |(x_1,y_2)| - |(x_2,y_1)|) + |(x_2,y_2)|$.

(37) For all elements x, y of $\mathcal{R}^n$ holds $|(x+y,x+y)| = |(x,x)| + 2\cdot|(x,y)| + |(y,y)|$.

(38) For all elements x, y of $\mathcal{R}^n$ holds $|(x-y,x-y)| = (|(x,x)| - 2\cdot|(x,y)|) + |(y,y)|$.

(39) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2\cdot|(x,y)| + |y|^{\mathbf{2}}$.

(40) For all elements x, y of $\mathcal{R}^n$ holds $|x-y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2\cdot|(x,y)|) + |y|^{\mathbf{2}}$.

(41) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} + |x-y|^{\mathbf{2}} = 2\cdot(|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(42) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} - |x-y|^{\mathbf{2}} = 4\cdot|(x,y)|$.

(43) For all elements x, y of $\mathcal{R}^n$ holds $||(x,y)|| \leq |x|\cdot|y|$.

(44) For all elements x, y of $\mathcal{R}^n$ holds $|x+y| \leq |x| + |y|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1, x_2 are orthogonal if and only if:

(Def. 6) $|(x_1,x_2)| = 0$.

Let us note that the predicate x_1, x_2 are orthogonal is symmetric.

Next we state the proposition

(45) Let R be a subset of $\mathbb{R}$ and x_1, x_2, y_1 be elements of $\mathcal{R}^n$. Suppose $R = \{|y_1-x|;x$ ranges over elements of $\mathcal{R}^n$: $x \in \mathrm{Line}(x_1,x_2)\}$. Then there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1,x_2)$ and $|y_1-y_2| = \inf R$ and x_1-x_2, y_1-y_2 are orthogonal.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{Line}(p_1,p_2)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 7) $\mathrm{Line}(p_1,p_2) = \{(1-l_1)\cdot p_1 + l_1\cdot p_2\}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. One can check that $\mathrm{Line}(p_1,p_2)$ is non empty.

In the sequel p_1, p_2, q_1, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$.

One can prove the following proposition

(46) $\mathrm{Line}(p_1,p_2) = \mathrm{Line}(p_2,p_1)$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us note that the functor $\mathrm{Line}(p_1, p_2)$ is commutative.

One can prove the following propositions:

(47) $p_1 \in \mathrm{Line}(p_1, p_2)$ and $p_2 \in \mathrm{Line}(p_1, p_2)$.

(48) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$, then $\mathrm{Line}(q_1, q_2) \subseteq \mathrm{Line}(p_1, p_2)$.

(49) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$ and $q_1 \neq q_2$, then $\mathrm{Line}(p_1, p_2) \subseteq \mathrm{Line}(q_1, q_2)$.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that A is line if and only if:

(Def. 8) There exist p_1, p_2 such that $p_1 \neq p_2$ and $A = \mathrm{Line}(p_1, p_2)$.

We introduce A is a line as a synonym of A is line.

Next we state three propositions:

(50) For all subsets A, C of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line and C is a line and $p_1 \in A$ and $p_2 \in A$ and $p_1 \in C$ and $p_2 \in C$ holds $p_1 = p_2$ or $A = C$.

(51) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exist p_1, p_2 such that $p_1 \in A$ and $p_2 \in A$ and $p_1 \neq p_2$.

(52) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exists p_2 such that $p_1 \neq p_2$ and $p_2 \in A$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{TPn2Rn}(p)$ yields an element of $\mathcal{R}^n$ and is defined by:

(Def. 9) $\mathrm{TPn2Rn}(p) = p$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|p|$ yields a real number and is defined as follows:

(Def. 10) $|p| = |\mathrm{TPn2Rn}(p)|$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p_1, p_2)|$ yields a real number and is defined as follows:

(Def. 11) $|(p_1, p_2)| = |(\mathrm{TPn2Rn}(p_1), \mathrm{TPn2Rn}(p_2))|$.

Let us notice that the functor $|(p_1, p_2)|$ is commutative.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that p_1, p_2 are orthogonal if and only if:

(Def. 12) $|(p_1, p_2)| = 0$.

Let us note that the predicate p_1, p_2 are orthogonal is symmetric.

One can prove the following proposition

(53) Let R be a subset of $\mathbb{R}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $R = \{|q_1 - p|; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $p \in \mathrm{Line}(p_1, p_2)\}$. Then there exists a point q_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $q_2 \in \mathrm{Line}(p_1, p_2)$ and $|q_1 - q_2| = \inf R$ and $p_1 - p_2$, $q_1 - q_2$ are orthogonal.

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