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The Euclidean Space

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Summary. The general definition of Euclidean Space.

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The articles [6], [16], [1], [14], [4], [11], [5], [10], [13], [2], [8], [7], [15], [9], [12], and [3] provide the notation and terminology for this paper.

In this paper k, n denote natural numbers and r denotes a real number.

Let us consider n. The functor $\mathcal{R}^n$ yields a non empty set of finite sequences of $\mathbb{R}$ and is defined as follows:

(Def. 1) $\mathcal{R}^n = \mathbb{R}^n$.

In the sequel x denotes a finite sequence of elements of $\mathbb{R}$.

The function $|\square|_{\mathbb{R}}$ from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 2) For every r holds $|\square|_{\mathbb{R}}(r) = |r|$.

Let us consider x. The functor $|x|$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 3) $|x| = |\square|_{\mathbb{R}} \cdot x$.

Let us consider n. The functor $\langle \underbrace{0, \ldots, 0}_{n} \rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 4) $\langle \underbrace{0, \ldots, 0}_{n} \rangle = n \mapsto (0$ **qua** real number$)$.

Let us consider n. Then $\langle \underbrace{0, \ldots, 0}_{n} \rangle$ is an element of $\mathcal{R}^n$.

In the sequel x, x_1, x_2, y denote elements of $\mathcal{R}^n$.

Let us consider n, x. Then $-x$ is an element of $\mathcal{R}^n$.

Let us consider n, x, y. Then $x + y$ is an element of $\mathcal{R}^n$. Then $x - y$ is an element of $\mathcal{R}^n$.

Let us consider n, let r be a real number, and let us consider x. Then $r \cdot x$ is an element of $\mathcal{R}^n$.

Let us consider n, x. Then $|x|$ is an element of $\mathbb{R}^n$.

Let us consider n, x. Then ${}^2x$ is an element of $\mathbb{R}^n$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $|x|$ yields a real number and is defined as follows:

(Def. 5) $|x| = \sqrt{\sum {}^2x}$.

One can prove the following propositions:

(2)[1] $\operatorname{len} x = n$.

(3) $\operatorname{dom} x = \operatorname{Seg} n$.

(4) $x(k) \in \mathbb{R}$.

(5) If for every k such that $k \in \operatorname{Seg} n$ holds $x_1(k) = x_2(k)$, then $x_1 = x_2$.

(6) If $r = x(k)$, then $|x|(k) = |r|$.

(7) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = n \mapsto (0$ **qua** real number$)$.

(8) $|-x| = |x|$.

(9) $|r \cdot x| = |r| \cdot |x|$.

(10) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = 0$.

(11) If $|x| = 0$, then $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(12) $|x| \geq 0$.

(13) $|-x| = |x|$.

(14) $|r \cdot x| = |r| \cdot |x|$.

(15) $|x_1 + x_2| \leq |x_1| + |x_2|$.

(16) $|x_1 - x_2| \leq |x_1| + |x_2|$.

(17) $|x_1| - |x_2| \leq |x_1 + x_2|$.

(18) $|x_1| - |x_2| \leq |x_1 - x_2|$.

(19) $|x_1 - x_2| = 0$ iff $x_1 = x_2$.

(20) If $x_1 \neq x_2$, then $|x_1 - x_2| > 0$.

(21) $|x_1 - x_2| = |x_2 - x_1|$.

(22) $|x_1 - x_2| \leq |x_1 - x| + |x - x_2|$.

Let us consider n. The functor ρ^n yielding a function from $[:\mathcal{R}^n, \mathcal{R}^n:]$ into $\mathbb{R}$ is defined by:

(Def. 6) For all elements x, y of $\mathcal{R}^n$ holds $\rho^n(x, y) = |x - y|$.

We now state two propositions:

(23) ${}^2(x - y) = {}^2(y - x)$.

(24) ρ^n is a metric of $\mathcal{R}^n$.

Let us consider n. The functor $\mathcal{E}^n$ yields a strict metric space and is defined as follows:

(Def. 7) $\mathcal{E}^n = \langle\mathcal{R}^n, \rho^n\rangle$.

Let us consider n. One can check that $\mathcal{E}^n$ is non empty.
Let us consider n. The functor $\mathcal{E}^n_{\mathrm{T}}$ yields a strict topological space and is defined by:

[1] The proposition (1) has been removed.

(Def. 8) $\mathcal{E}^n_{\rm T} = (\mathcal{E}^n)_{\rm top}$.

Let us consider n. Observe that $\mathcal{E}^n_{\rm T}$ is non empty.

We use the following convention: p, p_1, p_2, p_3 are points of $\mathcal{E}^n_{\rm T}$ and x, x_1, x_2, y, y_1, y_2 are real numbers.

The following four propositions are true:

(25) The carrier of $\mathcal{E}^n_{\rm T} = \mathcal{R}^n$.

(26) p is a function from $\mathrm{Seg}\, n$ into $\mathbb{R}$.

(27) p is a finite sequence of elements of $\mathbb{R}$.

(28) For every finite sequence f such that $f = p$ holds $\mathrm{len}\, f = n$.

Let us consider n. The functor $0_{\mathcal{E}^n_{\rm T}}$ yielding a point of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 9) $0_{\mathcal{E}^n_{\rm T}} = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

Let us consider n, p_1, p_2. The functor $p_1 + p_2$ yielding a point of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 10) For all elements p'_1, p'_2 of $\mathcal{R}^n$ such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 + p_2 = p'_1 + p'_2$.

Let us observe that the functor $p_1 + p_2$ is commutative.

One can prove the following propositions:

(29) For every element x of $\mathcal{R}^n$ holds ${}^2|x| = {}^2x$.

(30) $(p_1 + p_2) + p_3 = p_1 + (p_2 + p_3)$.

(31) $0_{\mathcal{E}^n_{\rm T}} + p = p$ and $p + 0_{\mathcal{E}^n_{\rm T}} = p$.

Let us consider x, n, p. The functor $x \cdot p$ yielding a point of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 11) For every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $x \cdot p = x \cdot p'$.

Next we state several propositions:

(32) $x \cdot 0_{\mathcal{E}^n_{\rm T}} = 0_{\mathcal{E}^n_{\rm T}}$.

(33) $1 \cdot p = p$ and $0 \cdot p = 0_{\mathcal{E}^n_{\rm T}}$.

(34) $(x \cdot y) \cdot p = x \cdot (y \cdot p)$.

(35) If $x \cdot p = 0_{\mathcal{E}^n_{\rm T}}$, then $x = 0$ or $p = 0_{\mathcal{E}^n_{\rm T}}$.

(36) $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.

(37) $(x + y) \cdot p = x \cdot p + y \cdot p$.

(38) If $x \cdot p_1 = x \cdot p_2$, then $x = 0$ or $p_1 = p_2$.

Let us consider n, p. The functor $-p$ yielding a point of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 12) For every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $-p = -p'$.

One can prove the following propositions:

(39) $--p = p$.

(40) $p + -p = 0_{\mathcal{E}^n_{\rm T}}$.

(41) If $p_1 + p_2 = 0_{\mathcal{E}^n_{\rm T}}$, then $p_1 = -p_2$ and $p_2 = -p_1$.

(42) $-(p_1 + p_2) = -p_1 + -p_2$.

(43) $-p = (-1) \cdot p$.

(44) $-x \cdot p = (-x) \cdot p$ and $-x \cdot p = x \cdot -p$.

Let us consider n, p_1, p_2. The functor $p_1 - p_2$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 13) For all elements p'_1, p'_2 of $\mathcal{R}^n$ such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 - p_2 = p'_1 - p'_2$.

The following propositions are true:

(45) $p_1 - p_2 = p_1 + -p_2$.

(46) $p - p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(47) If $p_1 - p_2 = 0_{\mathcal{E}^n_{\mathrm{T}}}$, then $p_1 = p_2$.

(48) $-(p_1 - p_2) = p_2 - p_1$ and $-(p_1 - p_2) = -p_1 + p_2$.

(49) $p_1 + (p_2 - p_3) = (p_1 + p_2) - p_3$.

(50) $p_1 - (p_2 + p_3) = p_1 - p_2 - p_3$.

(51) $p_1 - (p_2 - p_3) = (p_1 - p_2) + p_3$.

(52) $p = (p + p_1) - p_1$ and $p = (p - p_1) + p_1$.

(53) $x \cdot (p_1 - p_2) = x \cdot p_1 - x \cdot p_2$.

(54) $(x - y) \cdot p = x \cdot p - y \cdot p$.

In the sequel p, p_1, p_2 denote points of $\mathcal{E}^2_{\mathrm{T}}$.
The following proposition is true

(55) There exist real numbers x, y such that $p = \langle x, y \rangle$.

Let us consider p. The functor $p_{\mathbf{1}}$ yielding a real number is defined as follows:

(Def. 14) For every finite sequence f such that $p = f$ holds $p_{\mathbf{1}} = f(1)$.

The functor $p_{\mathbf{2}}$ yielding a real number is defined by:

(Def. 15) For every finite sequence f such that $p = f$ holds $p_{\mathbf{2}} = f(2)$.

Let x, y be real numbers. The functor $[x, y]$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 16) $[x, y] = \langle x, y \rangle$.

The following propositions are true:

(56) $[x, y]_{\mathbf{1}} = x$ and $[x, y]_{\mathbf{2}} = y$.

(57) $p = [p_{\mathbf{1}}, p_{\mathbf{2}}]$.

(58) $0_{\mathcal{E}^2_{\mathrm{T}}} = [0, 0]$.

(59) $p_1 + p_2 = [(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}, (p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}]$.

(60) $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

(61) $x \cdot p = [x \cdot p_{\mathbf{1}}, x \cdot p_{\mathbf{2}}]$.

(62) $x \cdot [x_1, y_1] = [x \cdot x_1, x \cdot y_1]$.

(63) $-p = [-p_{\mathbf{1}}, -p_{\mathbf{2}}]$.

(64) $-[x_1, y_1] = [-x_1, -y_1]$.

(65) $p_1 - p_2 = [(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}, (p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}]$.

(66) $[x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseqop.html`.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rvsum_1.html`.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[11] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/metric_1.html`.

[12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[13] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/absvalue.html`.

[14] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

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