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Equivalence Relations and Classes of Abstraction[1]

Konrad Raczkowski
Warsaw University
Białystok

Paweł Sadowski
Warsaw University
Białystok

Summary. In this article we deal with the notion of equivalence relation. The main properties of equivalence relations are proved. Then we define the classes of abstraction determined by an equivalence relation. Finally, the connections between a partition of a set and an equivalence relation are presented. We introduce the following notation of modes: *Equivalence Relation, a partition*.

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The articles [7], [5], [8], [9], [11], [10], [6], [2], [3], [1], and [4] provide the notation and terminology for this paper.

For simplicity, we use the following convention: X, Y, x, y, z denote sets, i, j denote natural numbers, A, B denote subsets of X, R, R_1, R_2 denote binary relations on X, and S_1 denotes a family of subsets of $[:X, X:]$.

One can prove the following proposition

(1) If $i < j$, then $j - i$ is a natural number.

Let us consider X. The functor ∇_X yielding a binary relation on X is defined as follows:

(Def. 1) $\nabla_X = [:X, X:]$.

Let us consider X. Note that ∇_X is total and reflexive.

Let us consider X and let us consider R_1, R_2. Then $R_1 \cap R_2$ is a binary relation on X. Then $R_1 \cup R_2$ is a binary relation on X.

The following proposition is true

(4)[1] id_X is reflexive in X and id_X is symmetric in X and id_X is transitive in X.

Let us consider X. A tolerance of X is a total reflexive symmetric binary relation on X. An equivalence relation of X is a total symmetric transitive binary relation on X.

One can prove the following propositions:

(6)[2] id_X is an equivalence relation of X.

(7) ∇_X is an equivalence relation of X.

Let us consider X. Note that ∇_X is total, symmetric, and transitive.

In the sequel E_1, E_2, E_3 are equivalence relations of X.

Next we state several propositions:

[1]Supported by RPBP.III-24.C8.

[1] The propositions (2) and (3) have been removed.

[2] The proposition (5) has been removed.

(11)[3] For every total reflexive binary relation R on X such that $x \in X$ holds $\langle x, x \rangle \in R$.

(12) For every total symmetric binary relation R on X such that $\langle x, y \rangle \in R$ holds $\langle y, x \rangle \in R$.

(13) For every total transitive binary relation R on X such that $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ holds $\langle x, z \rangle \in R$.

(14) For every total reflexive binary relation R on X such that there exists a set x such that $x \in X$ holds $R \neq \emptyset$.

(15) For every total binary relation R on X holds $\mathrm{field} R = X$.

(16) R is an equivalence relation of X iff R is reflexive, symmetric, and transitive and $\mathrm{field} R = X$.

Let us consider X and let us consider E_2, E_3. Then $E_2 \cap E_3$ is an equivalence relation of X.
We now state four propositions:

(17) $\mathrm{id}_X \cap E_1 = \mathrm{id}_X$.

(18) $\nabla_X \cap R = R$.

(19) Let given S_1. Suppose $S_1 \neq \emptyset$ and for every Y such that $Y \in S_1$ holds Y is an equivalence relation of X. Then $\bigcap S_1$ is an equivalence relation of X.

(20) For every R there exists E_1 such that $R \subseteq E_1$ and for every E_3 such that $R \subseteq E_3$ holds $E_1 \subseteq E_3$.

Let us consider X and let us consider E_2, E_3. The functor $E_2 \sqcup E_3$ yields an equivalence relation of X and is defined by:

(Def. 3)[4] $E_2 \cup E_3 \subseteq E_2 \sqcup E_3$ and for every E_1 such that $E_2 \cup E_3 \subseteq E_1$ holds $E_2 \sqcup E_3 \subseteq E_1$.

We now state two propositions:

(22)[5] $E_1 \sqcup E_1 = E_1$.

(23) $E_2 \sqcup E_3 = E_3 \sqcup E_2$.

Let us consider X and let us consider E_2, E_3. Let us note that the functor $E_2 \sqcup E_3$ is commutative.
Next we state two propositions:

(24) $E_2 \cap (E_2 \sqcup E_3) = E_2$.

(25) $E_2 \sqcup E_2 \cap E_3 = E_2$.

The scheme *Ex Eq Rel* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:
There exists an equivalence relation E_1 of $\mathcal{A}$ such that for all x, y holds $\langle x, y \rangle \in E_1$ iff $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x,y]$
provided the parameters meet the following requirements:

- For every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x,x]$,
- For all x, y such that $\mathcal{P}[x,y]$ holds $\mathcal{P}[y,x]$, and
- For all x, y, z such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$.

Let X be a set, let R be a tolerance of X, and let x be a set. The functor $[x]_R$ yields a subset of X and is defined by:

(Def. 4) $[x]_R = R^\circ\{x\}$.

We now state a number of propositions:

[3] The propositions (8)–(10) have been removed.
[4] The definition (Def. 2) has been removed.
[5] The proposition (21) has been removed.

(27)[6] For every tolerance R of X holds $y \in [x]_R$ iff $\langle y, x \rangle \in R$.

(28) For every tolerance R of X and for every x such that $x \in X$ holds $x \in [x]_R$.

(29) For every tolerance R of X and for every x such that $x \in X$ there exists y such that $x \in [y]_R$.

(30) For every transitive tolerance R of X such that $y \in [x]_R$ and $z \in [x]_R$ holds $\langle y, z \rangle \in R$.

(31) For every x such that $x \in X$ holds $y \in [x]_{(E_1)}$ iff $[x]_{(E_1)} = [y]_{(E_1)}$.

(32) For all x, y such that $x \in X$ and $y \in X$ holds $[x]_{(E_1)} = [y]_{(E_1)}$ or $[x]_{(E_1)}$ misses $[y]_{(E_1)}$.

(33) For every x such that $x \in X$ holds $[x]_{\mathrm{id}_X} = \{x\}$.

(34) For every x such that $x \in X$ holds $[x]_{\nabla_X} = X$.

(35) If there exists x such that $[x]_{(E_1)} = X$, then $E_1 = \nabla_X$.

(36) Suppose $x \in X$. Then $\langle x, y \rangle \in E_2 \sqcup E_3$ if and only if there exists a finite sequence f such that $1 \leq \mathrm{len}\, f$ and $x = f(1)$ and $y = f(\mathrm{len}\, f)$ and for every i such that $1 \leq i$ and $i < \mathrm{len}\, f$ holds $\langle f(i), f(i+1) \rangle \in E_2 \cup E_3$.

(37) For every equivalence relation E of X such that $E = E_2 \cup E_3$ and for every x such that $x \in X$ holds $[x]_E = [x]_{(E_2)}$ or $[x]_E = [x]_{(E_3)}$.

(38) If $E_2 \cup E_3 = \nabla_X$, then $E_2 = \nabla_X$ or $E_3 = \nabla_X$.

Let us consider X and let us consider E_1. The functor $\mathrm{Classes}\, E_1$ yields a family of subsets of X and is defined as follows:

(Def. 5) $A \in \mathrm{Classes}\, E_1$ iff there exists x such that $x \in X$ and $A = [x]_{(E_1)}$.

We now state the proposition

(40)[7] If $X = \emptyset$, then $\mathrm{Classes}\, E_1 = \emptyset$.

Let us consider X. A family of subsets of X is said to be a partition of X if:

(Def. 6)(i) $\bigcup \mathrm{it} = X$ and for every A such that $A \in \mathrm{it}$ holds $A \neq \emptyset$ and for every B such that $B \in \mathrm{it}$ holds $A = B$ or A misses B if $X \neq \emptyset$,

(ii) $\mathrm{it} = \emptyset$, otherwise.

The following propositions are true:

(42)[8] $\mathrm{Classes}\, E_1$ is a partition of X.

(43) For every partition P of X there exists E_1 such that $P = \mathrm{Classes}\, E_1$.

(44) For every x such that $x \in X$ holds $\langle x, y \rangle \in E_1$ iff $[x]_{(E_1)} = [y]_{(E_1)}$.

(45) If $x \in \mathrm{Classes}\, E_1$, then there exists an element y of X such that $x = [y]_{(E_1)}$.

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[6] The proposition (26) has been removed.
[7] The proposition (39) has been removed.
[8] The proposition (41) has been removed.

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