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A First-Order Predicate Calculus

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Summary. A continuation of [7], with an axiom system of first-order predicate theory. The consequence Cn of a set of formulas X is defined as the intersection of all theories containing X and some basic properties of it has been proved (monotonicity, idempotency, completeness etc.). The notion of a proof of given formula is also introduced and it is shown that $\mathrm{Cn}X = \{ p : p$ has a proof w.r.t. $X\}$. First 14 theorems are rather simple facts. I just wanted them to be included in the data base.

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The articles [10], [6], [12], [2], [13], [5], [3], [1], [8], [4], [11], [9], and [7] provide the notation and terminology for this paper.

In this paper n is a natural number.

Next we state several propositions:

(2)[1] If $n \leq 1$, then $n = 0$ or $n = 1$.

(3) If $n \leq 2$, then $n = 0$ or $n = 1$ or $n = 2$.

(4) If $n \leq 3$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$.

(5) If $n \leq 4$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$.

(6) If $n \leq 5$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$.

(7) If $n \leq 6$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$.

(8) If $n \leq 7$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$.

(9) If $n \leq 8$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$ or $n = 8$.

(10) If $n \leq 9$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$ or $n = 8$ or $n = 9$.

In the sequel i, j, n, k, l are natural numbers.

One can prove the following propositions:

(11) $\{k : k \leq n+1\} = \{i : i \leq n\} \cup \{n+1\}$.

(12) For every n holds $\{k : k \leq n\}$ is finite.

[1] The proposition (1) has been removed.

In the sequel X, Y, Z denote sets.
Next we state two propositions:

(13) If X is finite and $X \subseteq [:Y, Z:]$, then there exist sets A, B such that A is finite and $A \subseteq Y$ and B is finite and $B \subseteq Z$ and $X \subseteq [:A, B:]$.

(14) If X is finite and Z is finite and $X \subseteq [:Y, Z:]$, then there exists a set A such that A is finite and $A \subseteq Y$ and $X \subseteq [:A, Z:]$.

For simplicity, we adopt the following rules: T, S, X, Y denote subsets of CQC-WFF, p, q, r, t, F denote elements of CQC-WFF, s denotes a formula, and x, y denote bound variables.
Let us consider T. We say that T is a theory if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) VERUM $\in T$, and

(ii) for all p, q, r, s, x, y holds $(\neg p \Rightarrow p) \Rightarrow p \in T$ and $p \Rightarrow (\neg p \Rightarrow q) \in T$ and $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in T$ and $p \wedge q \Rightarrow q \wedge p \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$ and $\forall_x p \Rightarrow p \in T$ and if $p \Rightarrow q \in T$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in T$ and if $s(x) \in$ CQC-WFF and $s(y) \in$ CQC-WFF and $x \notin \mathrm{snb}(s)$ and $s(x) \in T$, then $s(y) \in T$.

We introduce T is a theory as a synonym of T is a theory.
We now state the proposition

(25)[2] If T is a theory and S is a theory, then $T \cap S$ is a theory.

Let us consider X. The functor $\mathrm{Cn}\,X$ yielding a subset of CQC-WFF is defined as follows:

(Def. 2) $t \in \mathrm{Cn}\,X$ iff for every T such that T is a theory and $X \subseteq T$ holds $t \in T$.

Next we state a number of propositions:

(27)[3] VERUM $\in \mathrm{Cn}\,X$.

(28) $(\neg p \Rightarrow p) \Rightarrow p \in \mathrm{Cn}\,X$.

(29) $p \Rightarrow (\neg p \Rightarrow q) \in \mathrm{Cn}\,X$.

(30) $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in \mathrm{Cn}\,X$.

(31) $p \wedge q \Rightarrow q \wedge p \in \mathrm{Cn}\,X$.

(32) If $p \in \mathrm{Cn}\,X$ and $p \Rightarrow q \in \mathrm{Cn}\,X$, then $q \in \mathrm{Cn}\,X$.

(33) $\forall_x p \Rightarrow p \in \mathrm{Cn}\,X$.

(34) If $p \Rightarrow q \in \mathrm{Cn}\,X$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in \mathrm{Cn}\,X$.

(35) If $s(x) \in$ CQC-WFF and $s(y) \in$ CQC-WFF and $x \notin \mathrm{snb}(s)$ and $s(x) \in \mathrm{Cn}\,X$, then $s(y) \in \mathrm{Cn}\,X$.

(36) $\mathrm{Cn}\,X$ is a theory.

(37) If T is a theory and $X \subseteq T$, then $\mathrm{Cn}\,X \subseteq T$.

(38) $X \subseteq \mathrm{Cn}\,X$.

(39) If $X \subseteq Y$, then $\mathrm{Cn}\,X \subseteq \mathrm{Cn}\,Y$.

(40) $\mathrm{Cn}\,\mathrm{Cn}\,X = \mathrm{Cn}\,X$.

(41) T is a theory iff $\mathrm{Cn}\,T = T$.

[2] The propositions (15)–(24) have been removed.
[3] The proposition (26) has been removed.

The set $\mathbb{K}$ is defined as follows:

(Def. 3) $\mathbb{K} = \{k : k \leq 9\}$.

Let us observe that $\mathbb{K}$ is non empty.
The following two propositions are true:

(43)[4] $0 \in \mathbb{K}$ and $1 \in \mathbb{K}$ and $2 \in \mathbb{K}$ and $3 \in \mathbb{K}$ and $4 \in \mathbb{K}$ and $5 \in \mathbb{K}$ and $6 \in \mathbb{K}$ and $7 \in \mathbb{K}$ and $8 \in \mathbb{K}$ and $9 \in \mathbb{K}$.

(44) $\mathbb{K}$ is finite.

In the sequel f, g are finite sequences of elements of [: CQC-WFF, $\mathbb{K}$:].
We now state the proposition

(45) Suppose $1 \leq n$ and $n \leq \operatorname{len} f$. Then $f(n)_{\mathbf{2}} = 0$ or $f(n)_{\mathbf{2}} = 1$ or $f(n)_{\mathbf{2}} = 2$ or $f(n)_{\mathbf{2}} = 3$ or $f(n)_{\mathbf{2}} = 4$ or $f(n)_{\mathbf{2}} = 5$ or $f(n)_{\mathbf{2}} = 6$ or $f(n)_{\mathbf{2}} = 7$ or $f(n)_{\mathbf{2}} = 8$ or $f(n)_{\mathbf{2}} = 9$.

Let P_1 be a finite sequence of elements of [: CQC-WFF, $\mathbb{K}$:] and let us consider n, X. We say that P_1 (n) is a correct proof step w.r.t. X if and only if:

(Def. 4)(i) $P_1(n)_{\mathbf{1}} \in X$ if $P_1(n)_{\mathbf{2}} = 0$,

(ii) $P_1(n)_{\mathbf{1}} = \mathrm{VERUM}$ if $P_1(n)_{\mathbf{2}} = 1$,

(iii) there exists p such that $P_1(n)_{\mathbf{1}} = (\neg p \Rightarrow p) \Rightarrow p$ if $P_1(n)_{\mathbf{2}} = 2$,

(iv) there exist p, q such that $P_1(n)_{\mathbf{1}} = p \Rightarrow (\neg p \Rightarrow q)$ if $P_1(n)_{\mathbf{2}} = 3$,

(v) there exist p, q, r such that $P_1(n)_{\mathbf{1}} = (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$ if $P_1(n)_{\mathbf{2}} = 4$,

(vi) there exist p, q such that $P_1(n)_{\mathbf{1}} = p \wedge q \Rightarrow q \wedge p$ if $P_1(n)_{\mathbf{2}} = 5$,

(vii) there exist p, x such that $P_1(n)_{\mathbf{1}} = \forall_x p \Rightarrow p$ if $P_1(n)_{\mathbf{2}} = 6$,

(viii) there exist i, j, p, q such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $p = P_1(j)_{\mathbf{1}}$ and $q = P_1(n)_{\mathbf{1}}$ and $P_1(i)_{\mathbf{1}} = p \Rightarrow q$ if $P_1(n)_{\mathbf{2}} = 7$,

(ix) there exist i, p, q, x such that $1 \leq i$ and $i < n$ and $P_1(i)_{\mathbf{1}} = p \Rightarrow q$ and $x \notin \operatorname{snb}(p)$ and $P_1(n)_{\mathbf{1}} = p \Rightarrow \forall_x q$ if $P_1(n)_{\mathbf{2}} = 8$,

(x) there exist i, x, y, s such that $1 \leq i$ and $i < n$ and $s(x) \in$ CQC-WFF and $s(y) \in$ CQC-WFF and $x \notin \operatorname{snb}(s)$ and $s(x) = P_1(i)_{\mathbf{1}}$ and $s(y) = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 9$.

Let us consider X, f. We say that f is a proof w.r.t. X if and only if:

(Def. 5) $f \neq \emptyset$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} f$ holds f (n) is a correct proof step w.r.t. X.

We now state several propositions:

(57)[5] If f is a proof w.r.t. X, then $\operatorname{rng} f \neq \emptyset$.

(58) If f is a proof w.r.t. X, then $1 \leq \operatorname{len} f$.

(59) If f is a proof w.r.t. X, then $f(1)_{\mathbf{2}} = 0$ or $f(1)_{\mathbf{2}} = 1$ or $f(1)_{\mathbf{2}} = 2$ or $f(1)_{\mathbf{2}} = 3$ or $f(1)_{\mathbf{2}} = 4$ or $f(1)_{\mathbf{2}} = 5$ or $f(1)_{\mathbf{2}} = 6$.

(60) Suppose $1 \leq n$ and $n \leq \operatorname{len} f$. Then f (n) is a correct proof step w.r.t. X if and only if $f \frown g$ (n) is a correct proof step w.r.t. X.

(61) Suppose $1 \leq n$ and $n \leq \operatorname{len} g$ and g (n) is a correct proof step w.r.t. X. Then $f \frown g$ ($n + \operatorname{len} f$) is a correct proof step w.r.t. X.

(62) If f is a proof w.r.t. X and g is a proof w.r.t. X, then $f \frown g$ is a proof w.r.t. X.

[4] The proposition (42) has been removed.
[5] The propositions (46)–(56) have been removed.

(63) If f is a proof w.r.t. X and $X \subseteq Y$, then f is a proof w.r.t. Y.

(64) If f is a proof w.r.t. X and $1 \leq l$ and $l \leq \operatorname{len} f$, then $f(l)_{\mathbf{1}} \in \operatorname{Cn} X$.

Let us consider f. Let us assume that $f \neq \emptyset$. The functor Effect f yields an element of CQC-WFF and is defined as follows:

(Def. 6) $\operatorname{Effect} f = f(\operatorname{len} f)_{\mathbf{1}}$.

Next we state several propositions:

(66)[6] If f is a proof w.r.t. X, then $\operatorname{Effect} f \in \operatorname{Cn} X$.

(67) $X \subseteq \{F : \bigvee_f\ (f \text{ is a proof w.r.t. } X \wedge \operatorname{Effect} f = F)\}$.

(68) For every X such that $Y = \{p : \bigvee_f\ (f \text{ is a proof w.r.t. } X \wedge \operatorname{Effect} f = p)\}$ holds Y is a theory.

(69) For every X holds $\{p : \bigvee_f\ (f \text{ is a proof w.r.t. } X \wedge \operatorname{Effect} f = p)\} = \operatorname{Cn} X$.

(70) $p \in \operatorname{Cn} X$ iff there exists f such that f is a proof w.r.t. X and $\operatorname{Effect} f = p$.

(71) If $p \in \operatorname{Cn} X$, then there exists Y such that $Y \subseteq X$ and Y is finite and $p \in \operatorname{Cn} Y$.

The subset Taut of CQC-WFF is defined as follows:

(Def. 8)[7] $\mathrm{Taut} = \operatorname{Cn}(\emptyset_{\text{CQC-WFF}})$.

We now state a number of propositions:

(74)[8] If T is a theory, then $\mathrm{Taut} \subseteq T$.

(75) $\mathrm{Taut} \subseteq \operatorname{Cn} X$.

(76) Taut is a theory.

(77) $\mathrm{VERUM} \in \mathrm{Taut}$.

(78) $(\neg p \Rightarrow p) \Rightarrow p \in \mathrm{Taut}$.

(79) $p \Rightarrow (\neg p \Rightarrow q) \in \mathrm{Taut}$.

(80) $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in \mathrm{Taut}$.

(81) $p \wedge q \Rightarrow q \wedge p \in \mathrm{Taut}$.

(82) If $p \in \mathrm{Taut}$ and $p \Rightarrow q \in \mathrm{Taut}$, then $q \in \mathrm{Taut}$.

(83) $\forall_x p \Rightarrow p \in \mathrm{Taut}$.

(84) If $p \Rightarrow q \in \mathrm{Taut}$ and $x \notin \operatorname{snb}(p)$, then $p \Rightarrow \forall_x q \in \mathrm{Taut}$.

(85) If $s(x) \in \text{CQC-WFF}$ and $s(y) \in \text{CQC-WFF}$ and $x \notin \operatorname{snb}(s)$ and $s(x) \in \mathrm{Taut}$, then $s(y) \in \mathrm{Taut}$.

Let us consider X, s. The predicate $X \vdash s$ is defined as follows:

(Def. 9) $s \in \operatorname{Cn} X$.

One can prove the following propositions:

(87)[9] $X \vdash \mathrm{VERUM}$.

[6] The proposition (65) has been removed.
[7] The definition (Def. 7) has been removed.
[8] The propositions (72) and (73) have been removed.
[9] The proposition (86) has been removed.

(88) $X \vdash (\neg p \Rightarrow p) \Rightarrow p$.

(89) $X \vdash p \Rightarrow (\neg p \Rightarrow q)$.

(90) $X \vdash (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$.

(91) $X \vdash p \wedge q \Rightarrow q \wedge p$.

(92) If $X \vdash p$ and $X \vdash p \Rightarrow q$, then $X \vdash q$.

(93) $X \vdash \forall_x p \Rightarrow p$.

(94) If $X \vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $X \vdash p \Rightarrow \forall_x q$.

(95) If $s(x) \in$ CQC-WFF and $s(y) \in$ CQC-WFF and $x \notin \mathrm{snb}(s)$ and $X \vdash s(x)$, then $X \vdash s(y)$.

Let us consider s. We say that s is valid if and only if:

(Def. 10) $\emptyset_{\text{CQC-WFF}} \vdash s$.

We introduce $\vdash s$ as a synonym of s is valid.

Let us consider s. Let us observe that s is valid if and only if:

(Def. 11) $s \in \mathrm{Taut}$.

Next we state a number of propositions:

(98)[10] If $\vdash p$, then $X \vdash p$.

(99) $\vdash$ VERUM.

(100) $\vdash (\neg p \Rightarrow p) \Rightarrow p$.

(101) $\vdash p \Rightarrow (\neg p \Rightarrow q)$.

(102) $\vdash (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$.

(103) $\vdash p \wedge q \Rightarrow q \wedge p$.

(104) If $\vdash p$ and $\vdash p \Rightarrow q$, then $\vdash q$.

(105) $\vdash \forall_x p \Rightarrow p$.

(106) If $\vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \forall_x q$.

(107) If $s(x) \in$ CQC-WFF and $s(y) \in$ CQC-WFF and $x \notin \mathrm{snb}(s)$ and $\vdash s(x)$, then $\vdash s(y)$.

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