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# Introduction to Concept Lattices

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**Summary.** In this paper we give Mizar formalization of concept lattices. Concept lattices stem from the so-called formal concept analysis — a part of applied mathematics that brings mathematical methods into the field of data analysis and knowledge processing. Our approach follows the one given in [8].

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The articles [11], [6], [14], [12], [15], [4], [3], [17], [16], [5], [13], [9], [7], [1], [10], and [2] provide the notation and terminology for this paper.

## 1. Formal Contexts

We introduce 2-sorted which are systems

⟨ objects, attributes ⟩,

where the objects and the attributes constitute sets.

Let $C$ be a 2-sorted. We say that $C$ is empty if and only if:

(Def. 1) The objects of $C$ are empty and the attributes of $C$ are empty.

We say that $C$ is quasi-empty if and only if:

(Def. 2) The objects of $C$ are empty or the attributes of $C$ are empty.

Let us observe that there exists a 2-sorted which is strict and non empty and there exists a 2-sorted which is strict and non quasi-empty.

Let us observe that there exists a 2-sorted which is strict, empty, and quasi-empty.

We consider ContextStr as extensions of 2-sorted as systems

⟨ objects, attributes, an information ⟩,

where the objects and the attributes constitute sets and the information is a relation between the objects and the attributes.

Let us note that there exists a ContextStr which is strict and non empty and there exists a ContextStr which is strict and non quasi-empty.

A FormalContext is a non quasi-empty ContextStr.

Let $C$ be a 2-sorted. An object of $C$ is an element of the objects of $C$. An Attribute of $C$ is an element of the attributes of $C$.

Let $C$ be a non quasi-empty 2-sorted. One can check that the attributes of $C$ is non empty and the objects of $C$ is non empty.

Let $C$ be a non quasi-empty 2-sorted. Observe that there exists a subset of the objects of $C$ which is non empty and there exists a subset of the attributes of $C$ which is non empty.

Let $C$ be a FormalContext, let $o$ be an object of $C$, and let $a$ be an Attribute of $C$. We say that $o$ is connected with $a$ if and only if:

 

(Def. 5)[1] $\langle o, a\rangle \in$ the information of $C$.

We introduce $o$ is not connected with $a$ as an antonym of $o$ is connected with $a$.

## 2. Derivation Operators

Let $C$ be a FormalContext. The functor ObjectDerivation$C$ yields a function from $2^{\text{the objects of } C}$ into $2^{\text{the attributes of } C}$ and is defined by the condition (Def. 6).

(Def. 6) Let $O$ be an element of $2^{\text{the objects of } C}$. Then $(\text{ObjectDerivation} C)(O) = \{a; a$ ranges over Attributes of $C$: $\bigwedge_{o\,:\,\text{object of } C} (o \in O \Rightarrow o$ is connected with $a)\}$.

Let $C$ be a FormalContext. The functor AttributeDerivation$C$ yielding a function from $2^{\text{the attributes of } C}$ into $2^{\text{the objects of } C}$ is defined by the condition (Def. 7).

(Def. 7) Let $A$ be an element of $2^{\text{the attributes of } C}$. Then $(\text{AttributeDerivation} C)(A) = \{o; o$ ranges over objects of $C$: $\bigwedge_{a\,:\,\text{Attribute of } C} (a \in A \Rightarrow o$ is connected with $a)\}$.

The following propositions are true:

(1) Let $C$ be a FormalContext and $o$ be an object of $C$. Then $(\text{ObjectDerivation} C)(\{o\}) = \{a; a$ ranges over Attributes of $C$: $o$ is connected with $a\}$.

(2) Let $C$ be a FormalContext and $a$ be an Attribute of $C$. Then $(\text{AttributeDerivation} C)(\{a\}) = \{o; o$ ranges over objects of $C$: $o$ is connected with $a\}$.

(3) For every FormalContext $C$ and for all subsets $O_1$, $O_2$ of the objects of $C$ such that $O_1 \subseteq O_2$ holds $(\text{ObjectDerivation} C)(O_2) \subseteq (\text{ObjectDerivation} C)(O_1)$.

(4) For every FormalContext $C$ and for all subsets $A_1, A_2$ of the attributes of $C$ such that $A_1 \subseteq A_2$ holds $(\text{AttributeDerivation} C)(A_2) \subseteq (\text{AttributeDerivation} C)(A_1)$.

(5) For every FormalContext $C$ and for every subset $O$ of the objects of $C$ holds $O \subseteq (\text{AttributeDerivation} C)((\text{ObjectDerivation} C)(O))$.

(6) For every FormalContext $C$ and for every subset $A$ of the attributes of $C$ holds $A \subseteq (\text{ObjectDerivation} C)((\text{AttributeDerivation} C)(A))$.

(7) For every FormalContext $C$ and for every subset $O$ of the objects of $C$ holds $(\text{ObjectDerivation} C)(O) = (\text{ObjectDerivation} C)((\text{AttributeDerivation} C)((\text{ObjectDerivation} C)(O)))$.

(8) For every FormalContext $C$ and for every subset $A$ of the attributes of $C$ holds $(\text{AttributeDerivation} C)(A) = (\text{AttributeDerivation} C)((\text{ObjectDerivation} C)((\text{AttributeDerivation} C)(A)))$.

(9) Let $C$ be a FormalContext, $O$ be a subset of the objects of $C$, and $A$ be a subset of the attributes of $C$. Then $O \subseteq (\text{AttributeDerivation} C)(A)$ if and only if $[: O, A :] \subseteq$ the information of $C$.

(10) Let $C$ be a FormalContext, $O$ be a subset of the objects of $C$, and $A$ be a subset of the attributes of $C$. Then $A \subseteq (\text{ObjectDerivation} C)(O)$ if and only if $[: O, A :] \subseteq$ the information of $C$.

(11) Let $C$ be a FormalContext, $O$ be a subset of the objects of $C$, and $A$ be a subset of the attributes of $C$. Then $O \subseteq (\text{AttributeDerivation} C)(A)$ if and only if $A \subseteq (\text{ObjectDerivation} C)(O)$.

Let $C$ be a FormalContext. The functor $\phi(C)$ yields a map from $2_{\subseteq}^{\text{the objects of } C}$ into $2_{\subseteq}^{\text{the attributes of } C}$ and is defined by:

(Def. 8) $\phi(C) = \text{ObjectDerivation} C$.

[1] The definitions (Def. 3) and (Def. 4) have been removed.

Let $C$ be a FormalContext. The functor psi$C$ yields a map from $2_{\subseteq}^{\text{the attributes of } C}$ into $2_{\subseteq}^{\text{the objects of } C}$ and is defined as follows:

(Def. 9) psi$C$ = AttributeDerivation$C$.

Let $P$, $R$ be non empty relational structures and let $C_1$ be a connection between $P$ and $R$. We say that $C_1$ is co-Galois if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exists a map $f$ from $P$ into $R$ and there exists a map $g$ from $R$ into $P$ such that

- (i) $C_1 = \langle f, g \rangle$,
- (ii) $f$ is antitone,
- (iii) $g$ is antitone, and
- (iv) for all elements $p_1$, $p_2$ of $P$ and for all elements $r_1$, $r_2$ of $R$ holds $p_1 \leq g(f(p_1))$ and $r_1 \leq f(g(r_1))$.

One can prove the following propositions:

(13)[2] Let $P$, $R$ be non empty posets, $C_1$ be a connection between $P$ and $R$, $f$ be a map from $P$ into $R$, and $g$ be a map from $R$ into $P$. Suppose $C_1 = \langle f, g \rangle$. Then $C_1$ is co-Galois if and only if for every element $p$ of $P$ and for every element $r$ of $R$ holds $p \leq g(r)$ iff $r \leq f(p)$.

(14) Let $P$, $R$ be non empty posets and $C_1$ be a connection between $P$ and $R$. Suppose $C_1$ is co-Galois. Let $f$ be a map from $P$ into $R$ and $g$ be a map from $R$ into $P$. If $C_1 = \langle f, g \rangle$, then $f = f \cdot (g \cdot f)$ and $g = g \cdot (f \cdot g)$.

(15) For every FormalContext $C$ holds $\langle \phi(C), \mathrm{psi}C \rangle$ is co-Galois.

(16) For every FormalContext $C$ and for all subsets $O_1$, $O_2$ of the objects of $C$ holds $(\text{ObjectDerivation}C)(O_1 \cup O_2) = (\text{ObjectDerivation}C)(O_1) \cap (\text{ObjectDerivation}C)(O_2)$.

(17) For every FormalContext $C$ and for all subsets $A_1$, $A_2$ of the attributes of $C$ holds $(\text{AttributeDerivation}C)(A_1 \cup A_2) = (\text{AttributeDerivation}C)(A_1) \cap (\text{AttributeDerivation}C)(A_2)$.

(18) For every FormalContext $C$ holds $(\text{ObjectDerivation}C)(\emptyset) =$ the attributes of $C$.

(19) For every FormalContext $C$ holds $(\text{AttributeDerivation}C)(\emptyset) =$ the objects of $C$.

## 3. Formal Concepts

Let $C$ be a 2-sorted. We consider ConceptStr over $C$ as systems

⟨ an extent, an intent ⟩,

where the extent is a subset of the objects of $C$ and the intent is a subset of the attributes of $C$.

Let $C$ be a 2-sorted and let $C_2$ be a ConceptStr over $C$. We say that $C_2$ is empty if and only if:

(Def. 11) The extent of $C_2$ is empty and the intent of $C_2$ is empty.

We say that $C_2$ is quasi-empty if and only if:

(Def. 12) The extent of $C_2$ is empty or the intent of $C_2$ is empty.

Let $C$ be a non quasi-empty 2-sorted. One can check that there exists a ConceptStr over $C$ which is strict and non empty and there exists a ConceptStr over $C$ which is strict and quasi-empty.

Let $C$ be an empty 2-sorted. One can check that every ConceptStr over $C$ is empty.

Let $C$ be a quasi-empty 2-sorted. One can check that every ConceptStr over $C$ is quasi-empty.

Let $C$ be a FormalContext and let $C_2$ be a ConceptStr over $C$. We say that $C_2$ is concept-like if and only if:

(Def. 13) $(\text{ObjectDerivation}C)(\text{the extent of } C_2) =$ the intent of $C_2$ and $(\text{AttributeDerivation}C)(\text{the intent of } C_2) =$ the extent of $C_2$.

[2] The proposition (12) has been removed.

Let $C$ be a FormalContext. Observe that there exists a ConceptStr over $C$ which is concept-like and non empty.

Let $C$ be a FormalContext. A FormalConcept of $C$ is a concept-like non empty ConceptStr over $C$.

Let $C$ be a FormalContext. Observe that there exists a FormalConcept of $C$ which is strict.

The following four propositions are true:

(20) Let $C$ be a FormalContext and $O$ be a subset of the objects of $C$. Then

(i) $\langle(\text{AttributeDerivation}\,C)((\text{ObjectDerivation}\,C)(O)),(\text{ObjectDerivation}\,C)(O)\rangle$ is a FormalConcept of $C$, and

(ii) for every subset $O'$ of the objects of $C$ and for every subset $A'$ of the attributes of $C$ such that $\langle O',A'\rangle$ is a FormalConcept of $C$ and $O \subseteq O'$ holds $(\text{AttributeDerivation}\,C)((\text{ObjectDerivation}\,C)(O)) \subseteq O'$.

(21) Let $C$ be a FormalContext and $O$ be a subset of the objects of $C$. Then there exists a subset $A$ of the attributes of $C$ such that $\langle O,A\rangle$ is a FormalConcept of $C$ if and only if $(\text{AttributeDerivation}\,C)((\text{ObjectDerivation}\,C)(O)) = O$.

(22) Let $C$ be a FormalContext and $A$ be a subset of the attributes of $C$. Then

(i) $\langle(\text{AttributeDerivation}\,C)(A),(\text{ObjectDerivation}\,C)((\text{AttributeDerivation}\,C)(A))\rangle$ is a FormalConcept of $C$, and

(ii) for every subset $O'$ of the objects of $C$ and for every subset $A'$ of the attributes of $C$ such that $\langle O',A'\rangle$ is a FormalConcept of $C$ and $A \subseteq A'$ holds $(\text{ObjectDerivation}\,C)((\text{AttributeDerivation}\,C)(A)) \subseteq A'$.

(23) Let $C$ be a FormalContext and $A$ be a subset of the attributes of $C$. Then there exists a subset $O$ of the objects of $C$ such that $\langle O,A\rangle$ is a FormalConcept of $C$ if and only if $(\text{ObjectDerivation}\,C)((\text{AttributeDerivation}\,C)(A)) = A$.

Let $C$ be a FormalContext and let $C_2$ be a ConceptStr over $C$. We say that $C_2$ is universal if and only if:

(Def. 14) The extent of $C_2 =$ the objects of $C$.

Let $C$ be a FormalContext and let $C_2$ be a ConceptStr over $C$. We say that $C_2$ is co-universal if and only if:

(Def. 15) The intent of $C_2 =$ the attributes of $C$.

Let $C$ be a FormalContext. Note that there exists a FormalConcept of $C$ which is strict and universal and there exists a FormalConcept of $C$ which is strict and co-universal.

Let $C$ be a FormalContext. The functor $\text{Concept} - \text{with} - \text{all} - \text{Objects}C$ yields a strict universal FormalConcept of $C$ and is defined by the condition (Def. 16).

(Def. 16) There exists a subset $O$ of the objects of $C$ and there exists a subset $A$ of the attributes of $C$ such that $\text{Concept} - \text{with} - \text{all} - \text{Objects}C = \langle O,A\rangle$ and $O = (\text{AttributeDerivation}\,C)(\emptyset)$ and $A = (\text{ObjectDerivation}\,C)((\text{AttributeDerivation}\,C)(\emptyset))$.

Let $C$ be a FormalContext. The functor $\text{Concept} - \text{with} - \text{all} - \text{Attributes}C$ yielding a strict co-universal FormalConcept of $C$ is defined by the condition (Def. 17).

(Def. 17) There exists a subset $O$ of the objects of $C$ and there exists a subset $A$ of the attributes of $C$ such that $\text{Concept} - \text{with} - \text{all} - \text{Attributes}C = \langle O,A\rangle$ and $O = (\text{AttributeDerivation}\,C)((\text{ObjectDerivation}\,C)(\emptyset))$ and $A = (\text{ObjectDerivation}\,C)(\emptyset)$.

Next we state several propositions:

(24) Let $C$ be a FormalContext. Then the extent of $\text{Concept} - \text{with} - \text{all} - \text{Objects}C =$ the objects of $C$ and the intent of $\text{Concept} - \text{with} - \text{all} - \text{Attributes}C =$ the attributes of $C$.

(25) Let $C$ be a FormalContext and $C_2$ be a FormalConcept of $C$. Then

(i) if the extent of $C_2 = \emptyset$, then $C_2$ is co-universal, and

(ii) if the intent of $C_2 = \emptyset$, then $C_2$ is universal.

(26) Let $C$ be a FormalContext and $C_2$ be a strict FormalConcept of $C$. Then

(i) if the extent of $C_2 = \emptyset$, then $C_2 = \mathrm{Concept-with-all-Attributes}C$, and

(ii) if the intent of $C_2 = \emptyset$, then $C_2 = \mathrm{Concept-with-all-Objects}C$.

(27) Let $C$ be a FormalContext and $C_2$ be a quasi-empty ConceptStr over $C$. Suppose $C_2$ is a FormalConcept of $C$. Then $C_2$ is universal and co-universal.

(28) Let $C$ be a FormalContext and $C_2$ be a quasi-empty ConceptStr over $C$. If $C_2$ is a strict FormalConcept of $C$, then $C_2 = \mathrm{Concept-with-all-Objects}C$ or $C_2 = \mathrm{Concept-with-all-Attributes}C$.

Let $C$ be a FormalContext. A non empty set is called a Set of FormalConcepts of $C$ if:

(Def. 18) For every set $X$ such that $X \in$ it holds $X$ is a FormalConcept of $C$.

Let $C$ be a FormalContext and let $F_1$ be a Set of FormalConcepts of $C$. We see that the element of $F_1$ is a FormalConcept of $C$.

Let $C$ be a FormalContext and let $C_3$, $C_4$ be FormalConcepts of $C$. We say that $C_3$ is SubConcept of $C_4$ if and only if:

(Def. 19) The extent of $C_3 \subseteq$ the extent of $C_4$.

We introduce $C_4$ is SuperConcept of $C_3$ as a synonym of $C_3$ is SubConcept of $C_4$.

Next we state three propositions:

(31)[3] Let $C$ be a FormalContext and $C_3$, $C_4$ be FormalConcepts of $C$. Then $C_3$ is SubConcept of $C_4$ if and only if the intent of $C_4 \subseteq$ the intent of $C_3$.

(33)[4] Let $C$ be a FormalContext and $C_3$, $C_4$ be FormalConcepts of $C$. Then $C_3$ is SuperConcept of $C_4$ if and only if the intent of $C_3 \subseteq$ the intent of $C_4$.

(34) Let $C$ be a FormalContext and $C_2$ be a FormalConcept of $C$. Then $C_2$ is SubConcept of $\mathrm{Concept-with-all-Objects}C$ and $\mathrm{Concept-with-all-Attributes}C$ is SubConcept of $C_2$.

## 4. Concept Lattices

Let $C$ be a FormalContext. The functor $\mathrm{B-carrier}C$ yields a non empty set and is defined by the condition (Def. 20).

(Def. 20) $\mathrm{B-carrier}C = \{\langle E, I\rangle; E$ ranges over subsets of the objects of $C$, $I$ ranges over subsets of the attributes of $C$: $\langle E, I\rangle$ is non empty $\wedge$ $(\mathrm{ObjectDerivation}C)(E) = I$ $\wedge$ $(\mathrm{AttributeDerivation}C)(I) = E\}$.

Let $C$ be a FormalContext. Then $\mathrm{B-carrier}C$ is a Set of FormalConcepts of $C$.

Let $C$ be a FormalContext. Observe that $\mathrm{B-carrier}C$ is non empty.

We now state the proposition

(35) For every FormalContext $C$ and for every set $C_2$ holds $C_2 \in \mathrm{B-carrier}C$ iff $C_2$ is a strict FormalConcept of $C$.

Let $C$ be a FormalContext. The functor $\mathrm{B-meet}C$ yields a binary operation on $\mathrm{B-carrier}C$ and is defined by the condition (Def. 21).

[3] The propositions (29) and (30) have been removed.

[4] The proposition (32) has been removed.

(Def. 21) Let $C_3$, $C_4$ be strict FormalConcepts of $C$. Then there exists a subset $O$ of the objects of $C$ and there exists a subset $A$ of the attributes of $C$ such that $(\mathrm{B}-\mathrm{meet}C)(C_3, C_4) = \langle O, A\rangle$ and $O = (\text{the extent of } C_3) \cap (\text{the extent of } C_4)$ and $A = (\mathrm{ObjectDerivation}C)((\mathrm{AttributeDerivation}C)((\text{the intent of } C_3) \cup (\text{the intent of } C_4)))$.

Let $C$ be a FormalContext. The functor $\mathrm{B}-\mathrm{join}C$ yields a binary operation on $\mathrm{B}-\mathrm{carrier}C$ and is defined by the condition (Def. 22).

(Def. 22) Let $C_3$, $C_4$ be strict FormalConcepts of $C$. Then there exists a subset $O$ of the objects of $C$ and there exists a subset $A$ of the attributes of $C$ such that $(\mathrm{B}-\mathrm{join}C)(C_3, C_4) = \langle O, A\rangle$ and $O = (\mathrm{AttributeDerivation}C)((\mathrm{ObjectDerivation}C)((\text{the extent of } C_3) \cup (\text{the extent of } C_4)))$ and $A = (\text{the intent of } C_3) \cap (\text{the intent of } C_4)$.

The following propositions are true:

(36) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$ of $C$ holds $(\mathrm{B}-\mathrm{meet}C)(C_3, C_4) = (\mathrm{B}-\mathrm{meet}C)(C_4, C_3)$.

(37) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$ of $C$ holds $(\mathrm{B}-\mathrm{join}C)(C_3, C_4) = (\mathrm{B}-\mathrm{join}C)(C_4, C_3)$.

(38) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$, $C_5$ of $C$ holds $(\mathrm{B}-\mathrm{meet}C)(C_3, (\mathrm{B}-\mathrm{meet}C)(C_4, C_5)) = (\mathrm{B}-\mathrm{meet}C)((\mathrm{B}-\mathrm{meet}C)(C_3, C_4), C_5)$.

(39) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$, $C_5$ of $C$ holds $(\mathrm{B}-\mathrm{join}C)(C_3, (\mathrm{B}-\mathrm{join}C)(C_4, C_5)) = (\mathrm{B}-\mathrm{join}C)((\mathrm{B}-\mathrm{join}C)(C_3, C_4), C_5)$.

(40) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$ of $C$ holds $(\mathrm{B}-\mathrm{join}C)((\mathrm{B}-\mathrm{meet}C)(C_3, C_4), C_4) = C_4$.

(41) For every FormalContext $C$ and for all strict FormalConcepts $C_3$, $C_4$ of $C$ holds $(\mathrm{B}-\mathrm{meet}C)(C_3, (\mathrm{B}-\mathrm{join}C)(C_3, C_4)) = C_3$.

(42) For every FormalContext $C$ and for every strict FormalConcept $C_2$ of $C$ holds $(\mathrm{B}-\mathrm{meet}C)(C_2, \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}C) = C_2$.

(43) For every FormalContext $C$ and for every strict FormalConcept $C_2$ of $C$ holds $(\mathrm{B}-\mathrm{join}C)(C_2, \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}C) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}C$.

(44) For every FormalContext $C$ and for every strict FormalConcept $C_2$ of $C$ holds $(\mathrm{B}-\mathrm{join}C)(C_2, \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}C) = C_2$.

(45) For every FormalContext $C$ and for every strict FormalConcept $C_2$ of $C$ holds $(\mathrm{B}-\mathrm{meet}C)(C_2, \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}C) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}C$.

Let $C$ be a FormalContext. The functor $\mathrm{ConceptLattice}C$ yielding a strict non empty lattice structure is defined as follows:

(Def. 23) $\mathrm{ConceptLattice}C = \langle \mathrm{B}-\mathrm{carrier}C, \mathrm{B}-\mathrm{join}C, \mathrm{B}-\mathrm{meet}C\rangle$.

Next we state the proposition

(46) For every FormalContext $C$ holds $\mathrm{ConceptLattice}C$ is a lattice.

Let $C$ be a FormalContext. One can verify that $\mathrm{ConceptLattice}C$ is strict, non empty, and lattice-like.

Let $C$ be a FormalContext and let $S$ be a non empty subset of $\mathrm{ConceptLattice}C$. We see that the element of $S$ is a FormalConcept of $C$.

Let $C$ be a FormalContext and let $C_2$ be an element of $\mathrm{ConceptLattice}C$. The functor ${C_2}^{\mathrm{T}}$ yielding a strict FormalConcept of $C$ is defined as follows:

(Def. 24) ${C_2}^{\mathrm{T}} = C_2$.

We now state two propositions:

(47) For every FormalContext $C$ and for all elements $C_3$, $C_4$ of ConceptLattice$C$ holds $C_3 \sqsubseteq C_4$ iff ${C_3}^{\mathrm{T}}$ is SubConcept of ${C_4}^{\mathrm{T}}$.

(48) For every FormalContext $C$ holds ConceptLattice$C$ is a complete lattice.

Let $C$ be a FormalContext. One can verify that ConceptLattice$C$ is complete.

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