JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Conjugate Sequences, Bounded Complex Sequences and Convergent Complex Sequences

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Summary. This article is a continuation of [1]. It is divided into five sections. The first one contains a few useful lemmas. In the second part there is a definition of conjugate sequences and proofs of some basic properties of such sequences. The third segment treats of bounded complex sequences,next one contains description of convergent complex sequences. The last and the biggest part of the article contains proofs of main theorems concerning the theory of bounded and convergent complex sequences.

MML Identifier: COMSEQ_2.

WWW: http://mizar.org/JFM/Vol8/comseq_2.html

The articles [10], [2], [8], [6], [3], [11], [4], [7], [9], [5], and [1] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following rules: n, m denote natural numbers, r, g denote elements of $\mathbb{C}$, and s, s', s_1 denote complex sequences.

We now state two propositions:

(1) If $g \neq 0_{\mathbb{C}}$ and $r \neq 0_{\mathbb{C}}$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g| \cdot |r|}$.

(2) For every n there exists a real number r such that $0 < r$ and for every m such that $m \leq n$ holds $|s(m)| < r$.

2. Conjugate sequences

Let C be a non empty set and let f be a partial function from C to $\mathbb{C}$. The functor $\overline{f}$ yields a partial function from C to $\mathbb{C}$ and is defined as follows:

(Def. 1) $\operatorname{dom} \overline{f} = \operatorname{dom} f$ and for every element c of C such that $c \in \operatorname{dom} \overline{f}$ holds $\overline{f}(c) = \overline{f_c}$.

Let C be a non empty set and let f be a function from C into $\mathbb{C}$. Then $\overline{f}$ can be characterized by the condition:

(Def. 2) $\operatorname{dom} \overline{f} = C$ and for every element n of C holds $\overline{f}(n) = \overline{f(n)}$.

Let C be a non empty set and let s_2 be a function from C into $\mathbb{C}$. Note that $\overline{s_2}$ is total.

The following propositions are true:

(3) If s is non-zero, then $\overline{s}$ is non-zero.

(4) $\overline{r\,s} = \overline{r}\,\overline{s}$.

(5) $\overline{s\,s'} = \overline{s}\,\overline{s'}$.

(6) $\overline{s}^{-1} = \overline{s^{-1}}$.

(7) $\overline{s'/s} = \overline{s'}/\overline{s}$.

3. BOUNDED COMPLEX SEQUENCES

Let us consider s. We say that s is bounded if and only if:

(Def. 3) There exists a real number r such that for every n holds $|s(n)| < r$.

Let us note that there exists a complex sequence which is bounded.
The following proposition is true

(8) s is bounded iff there exists a real number r such that $0 < r$ and for every n holds $|s(n)| < r$.

4. CONVERGENT COMPLEX SEQUENCES

Let us consider s. We say that s is convergent if and only if:

(Def. 4) There exists g such that for every real number p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|s(m) - g| < p$.

Let us consider s. Let us assume that s is convergent. The functor $\lim s$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 5) For every real number p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|s(m) - \lim s| < p$.

The following two propositions are true:

(9) If there exists g such that for every natural number n holds $s(n) = g$, then s is convergent.

(10) For every g such that for every natural number n holds $s(n) = g$ holds $\lim s = g$.

Let us note that there exists a complex sequence which is convergent.
Let s be a convergent complex sequence. Observe that $|s|$ is convergent.
One can prove the following proposition

(11) If s is convergent, then $\lim |s| = |\lim s|$.

Let s be a convergent complex sequence. One can verify that $\overline{s}$ is convergent.
We now state the proposition

(12) If s is convergent, then $\lim \overline{s} = \overline{\lim s}$.

5. MAIN THEOREMS

Next we state a number of propositions:

(13) If s is convergent and s' is convergent, then $s + s'$ is convergent.

(14) If s is convergent and s' is convergent, then $\lim(s + s') = \lim s + \lim s'$.

(15) If s is convergent and s' is convergent, then $\lim |s + s'| = |\lim s + \lim s'|$.

(16) If s is convergent and s' is convergent, then $\lim \overline{s + s'} = \overline{\lim s} + \overline{\lim s'}$.

(17) If s is convergent, then $r\,s$ is convergent.

(18) If s is convergent, then $\lim(r\,s) = r \cdot \lim s$.

(19) If s is convergent, then $\lim |r\,s| = |r| \cdot |\lim s|$.

(20) If s is convergent, then $\lim \overline{r\,s} = \overline{r} \cdot \overline{\lim s}$.

(21) If s is convergent, then $-s$ is convergent.

(22) If s is convergent, then $\lim(-s) = -\lim s$.

(23) If s is convergent, then $\lim |-s| = |\lim s|$.

(24) If s is convergent, then $\lim \overline{-s} = -\overline{\lim s}$.

(25) If s is convergent and s' is convergent, then $s - s'$ is convergent.

(26) If s is convergent and s' is convergent, then $\lim(s - s') = \lim s - \lim s'$.

(27) If s is convergent and s' is convergent, then $\lim |s - s'| = |\lim s - \lim s'|$.

(28) If s is convergent and s' is convergent, then $\lim \overline{s - s'} = \overline{\lim s} - \overline{\lim s'}$.

Let us note that every complex sequence which is convergent is also bounded.

One can check that every complex sequence which is non bounded is also non convergent.

We now state a number of propositions:

(29) If s is a convergent complex sequence and s' is a convergent complex sequence, then $s\,s'$ is convergent.

(30) If s is a convergent complex sequence and s' is a convergent complex sequence, then $\lim(s\,s') = \lim s \cdot \lim s'$.

(31) If s is convergent and s' is convergent, then $\lim |s\,s'| = |\lim s| \cdot |\lim s'|$.

(32) If s is convergent and s' is convergent, then $\lim \overline{s\,s'} = \overline{\lim s} \cdot \overline{\lim s'}$.

(33) If s is convergent, then if $\lim s \neq 0_{\mathbb{C}}$, then there exists n such that for every m such that $n \leq m$ holds $\frac{|\lim s|}{2} < |s(m)|$.

(34) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then s^{-1} is convergent.

(35) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(s^{-1}) = (\lim s)^{-1}$.

(36) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |s^{-1}| = |\lim s|^{-1}$.

(37) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim \overline{s^{-1}} = \overline{\lim s}^{-1}$.

(38) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then s'/s is convergent.

(39) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(s'/s) = \frac{\lim s'}{\lim s}$.

(40) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |s'/s| = \frac{|\lim s'|}{|\lim s|}$.

(41) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim \overline{s'/s} = \overline{\frac{\lim s'}{\lim s}}$.

(42) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $s\,s_1$ is convergent.

(43) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim(s\,s_1) = 0_{\mathbb{C}}$.

(44) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim |s\,s_1| = 0$.

(45) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim \overline{s\,s_1} = 0_{\mathbb{C}}$.

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Received December 20, 1996

Published January 2, 2004