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# Trigonometric Form of Complex Numbers

Robert Milewski
University of Białystok

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The articles [16], [21], [1], [17], [6], [18], [12], [13], [19], [9], [2], [22], [3], [8], [14], [7], [4], [5], [15], [20], [23], [10], and [11] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

The scheme *Regr without 0* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[1]$

provided the parameters meet the following requirements:

- There exists a non empty natural number $k$ such that $\mathcal{P}[k]$, and
- For every non empty natural number $k$ such that $k \neq 1$ and $\mathcal{P}[k]$ there exists a non empty natural number $n$ such that $n < k$ and $\mathcal{P}[n]$.

The following propositions are true:

(3)[1] For every element $z$ of $\mathbb{C}$ holds $\Re(z) \geq -|z|$.

(4) For every element $z$ of $\mathbb{C}$ holds $\Im(z) \geq -|z|$.

(5) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\Re(z) \geq -|z|$.

(6) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\Im(z) \geq -|z|$.

(7) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $|z|^{\mathbf{2}} = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$.

(8) For all real numbers $x_1$, $x_2$, $y_1$, $y_2$ such that $x_1 + x_2 i_{\mathbb{C}_{\rm F}} = y_1 + y_2 i_{\mathbb{C}_{\rm F}}$ holds $x_1 = y_1$ and $x_2 = y_2$.

(9) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $z = \Re(z) + \Im(z) i_{\mathbb{C}_{\rm F}}$.

(10) $0_{\mathbb{C}_{\rm F}} = 0 + 0 i_{\mathbb{C}_{\rm F}}$.

(12)[2] For every unital non empty groupoid $L$ and for every element $x$ of $L$ holds $\mathrm{power}_L(x, 1) = x$.

(13) For every unital non empty groupoid $L$ and for every element $x$ of $L$ holds $\mathrm{power}_L(x, 2) = x \cdot x$.

(14) Let $L$ be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and $n$ be a natural number. If $n > 0$, then $\mathrm{power}_L(0_L, n) = 0_L$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (11) has been removed.

(15) Let $L$ be an associative commutative unital non empty groupoid, $x$, $y$ be elements of $L$, and $n$ be a natural number. Then $\mathrm{power}_L(x \cdot y, n) = \mathrm{power}_L(x, n) \cdot \mathrm{power}_L(y, n)$.

(16) For every real number $x$ such that $x > 0$ and for every natural number $n$ holds $\mathrm{power}_{\mathbb{C}_\mathrm{F}}(x + 0i_{\mathbb{C}_\mathrm{F}}, n) = x^n + 0i_{\mathbb{C}_\mathrm{F}}$.

(17) For every real number $x$ and for every natural number $n$ such that $x \geq 0$ and $n \neq 0$ holds $\sqrt[n]{x^n} = x$.

## 2. Sinus and Cosinus Properties

Next we state several propositions:

(20)[3] $\pi + \frac{\pi}{2} = \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi + \frac{\pi}{2} = 2 \cdot \pi$ and $\frac{3}{2} \cdot \pi - \pi = \frac{\pi}{2}$.

(21) $0 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$ and $0 < \pi$ and $-\frac{\pi}{2} < \frac{\pi}{2}$ and $\pi < 2 \cdot \pi$ and $\frac{\pi}{2} < \frac{3}{2} \cdot \pi$ and $-\frac{\pi}{2} < 0$ and $0 < 2 \cdot \pi$ and $\pi < \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi < 2 \cdot \pi$ and $0 < \frac{3}{2} \cdot \pi$.

(22) For all real numbers $a$, $b$, $c$, $x$ such that $x \in ]a, c[$ holds $x \in ]a, b[$ or $x = b$ or $x \in ]b, c[$.

(23) For every real number $x$ such that $x \in ]0, \pi[$ holds $\sin(x) > 0$.

(24) For every real number $x$ such that $x \in [0, \pi]$ holds $\sin(x) \geq 0$.

(25) For every real number $x$ such that $x \in ]\pi, 2 \cdot \pi[$ holds $\sin(x) < 0$.

(26) For every real number $x$ such that $x \in [\pi, 2 \cdot \pi]$ holds $\sin(x) \leq 0$.

In the sequel $x$ is a real number.
One can prove the following propositions:

(27) If $x \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$, then $\cos(x) > 0$.

(28) For every real number $x$ such that $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ holds $\cos(x) \geq 0$.

(29) If $x \in ]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$, then $\cos(x) < 0$.

(30) For every real number $x$ such that $x \in [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ holds $\cos(x) \leq 0$.

(31) If $x \in ]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$, then $\cos(x) > 0$.

(32) For every real number $x$ such that $x \in [\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ holds $\cos(x) \geq 0$.

(33) For every real number $x$ such that $0 \leq x$ and $x < 2 \cdot \pi$ and $\sin x = 0$ holds $x = 0$ or $x = \pi$.

(34) For every real number $x$ such that $0 \leq x$ and $x < 2 \cdot \pi$ and $\cos x = 0$ holds $x = \frac{\pi}{2}$ or $x = \frac{3}{2} \cdot \pi$.

(35) sin is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(36) sin is decreasing on $]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$.

(37) cos is decreasing on $]0, \pi[$.

(38) cos is increasing on $]\pi, 2 \cdot \pi[$.

(39) sin is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(40) sin is decreasing on $[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$.

(41) cos is decreasing on $[0, \pi]$.

(42) cos is increasing on $[\pi, 2 \cdot \pi]$.

[3] The propositions (18) and (19) have been removed.

(43) sin is continuous on $\mathbb{R}$ and for all real numbers $x$, $y$ holds sin is continuous on $[x,y]$ and sin is continuous on $]x,y[$.

(44) cos is continuous on $\mathbb{R}$ and for all real numbers $x$, $y$ holds cos is continuous on $[x,y]$ and cos is continuous on $]x,y[$.

(45) $\sin(x) \in [-1,1]$ and $\cos(x) \in [-1,1]$.

(46) $\mathrm{rng}\sin = [-1,1]$.

(47) $\mathrm{rng}\cos = [-1,1]$.

(48) $\mathrm{rng}(\sin\restriction[-\frac{\pi}{2},\frac{\pi}{2}]) = [-1,1]$.

(49) $\mathrm{rng}(\sin\restriction[\frac{\pi}{2},\frac{3}{2}\cdot\pi]) = [-1,1]$.

(50) $\mathrm{rng}(\cos\restriction[0,\pi]) = [-1,1]$.

(51) $\mathrm{rng}(\cos\restriction[\pi,2\cdot\pi]) = [-1,1]$.

## 3. Argument of Complex Number

Let $z$ be an element of $\mathbb{C}_{\rm F}$. The functor $\mathrm{Arg}\,z$ yields a real number and is defined by:

(Def. 1)(i) $z = |z|\cdot\cos\mathrm{Arg}\,z + (|z|\cdot\sin\mathrm{Arg}\,z)i_{\mathbb{C}_{\rm F}}$ and $0 \leq \mathrm{Arg}\,z$ and $\mathrm{Arg}\,z < 2\cdot\pi$ if $z \neq 0_{\mathbb{C}_{\rm F}}$,

(ii) $\mathrm{Arg}\,z = 0$, otherwise.

One can prove the following propositions:

(52) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $0 \leq \mathrm{Arg}\,z$ and $\mathrm{Arg}\,z < 2\cdot\pi$.

(53) For every real number $x$ such that $x \geq 0$ holds $\mathrm{Arg}(x+0i_{\mathbb{C}_{\rm F}}) = 0$.

(54) For every real number $x$ such that $x < 0$ holds $\mathrm{Arg}(x+0i_{\mathbb{C}_{\rm F}}) = \pi$.

(55) For every real number $x$ such that $x > 0$ holds $\mathrm{Arg}(0+xi_{\mathbb{C}_{\rm F}}) = \frac{\pi}{2}$.

(56) For every real number $x$ such that $x < 0$ holds $\mathrm{Arg}(0+xi_{\mathbb{C}_{\rm F}}) = \frac{3}{2}\cdot\pi$.

(57) $\mathrm{Arg}(\mathbf{1}_{\mathbb{C}_{\rm F}}) = 0$.

(58) $\mathrm{Arg}\,i_{\mathbb{C}_{\rm F}} = \frac{\pi}{2}$.

(59) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\mathrm{Arg}\,z \in ]0,\frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(60) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\mathrm{Arg}\,z \in ]\frac{\pi}{2},\pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(61) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\mathrm{Arg}\,z \in ]\pi,\frac{3}{2}\cdot\pi[$ iff $\Re(z) < 0$ and $\Im(z) < 0$.

(62) For every element $z$ of $\mathbb{C}_{\rm F}$ holds $\mathrm{Arg}\,z \in ]\frac{3}{2}\cdot\pi,2\cdot\pi[$ iff $\Re(z) > 0$ and $\Im(z) < 0$.

(63) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Im(z) > 0$ holds $\sin\mathrm{Arg}\,z > 0$.

(64) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Im(z) < 0$ holds $\sin\mathrm{Arg}\,z < 0$.

(65) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Im(z) \geq 0$ holds $\sin\mathrm{Arg}\,z \geq 0$.

(66) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Im(z) \leq 0$ holds $\sin\mathrm{Arg}\,z \leq 0$.

(67) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Re(z) > 0$ holds $\cos\mathrm{Arg}\,z > 0$.

(68) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Re(z) < 0$ holds $\cos\mathrm{Arg}\,z < 0$.

(69) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Re(z) \geq 0$ holds $\cos\mathrm{Arg}\,z \geq 0$.

(70) For every element $z$ of $\mathbb{C}_{\rm F}$ such that $\Re(z) \leq 0$ and $z \neq 0_{\mathbb{C}_{\rm F}}$ holds $\cos \operatorname{Arg} z \leq 0$.

(71) For every natural number $n$ holds $\mathrm{power}_{\mathbb{C}_{\rm F}}(\cos x + \sin x i_{\mathbb{C}_{\rm F}}, n) = \cos(n \cdot x) + \sin(n \cdot x) i_{\mathbb{C}_{\rm F}}$.

(72) Let $z$ be an element of $\mathbb{C}_{\rm F}$ and $n$ be a natural number. If $z \neq 0_{\mathbb{C}_{\rm F}}$ or $n \neq 0$, then $\mathrm{power}_{\mathbb{C}_{\rm F}}(z, n) = |z|^n \cdot \cos(n \cdot \operatorname{Arg} z) + (|z|^n \cdot \sin(n \cdot \operatorname{Arg} z)) i_{\mathbb{C}_{\rm F}}$.

(73) For all natural numbers $n$, $k$ such that $n \neq 0$ holds $\mathrm{power}_{\mathbb{C}_{\rm F}}(\cos(\frac{x+2\cdot\pi\cdot k}{n}) + \sin(\frac{x+2\cdot\pi\cdot k}{n}) i_{\mathbb{C}_{\rm F}}, n) = \cos x + \sin x i_{\mathbb{C}_{\rm F}}$.

(74) Let $z$ be an element of $\mathbb{C}_{\rm F}$ and $n$, $k$ be natural numbers. If $n \neq 0$, then $z = \mathrm{power}_{\mathbb{C}_{\rm F}}(\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n})) i_{\mathbb{C}_{\rm F}}, n)$.

Let $x$ be an element of $\mathbb{C}_{\rm F}$ and let $n$ be a non empty natural number. An element of $\mathbb{C}_{\rm F}$ is called a complex root of $n$, $x$ if:

(Def. 2) $\mathrm{power}_{\mathbb{C}_{\rm F}}(\mathrm{it}, n) = x$.

One can prove the following four propositions:

(75) Let $x$ be an element of $\mathbb{C}_{\rm F}$, $n$ be a non empty natural number, and $k$ be a natural number. Then $\sqrt[n]{|x|} \cdot \cos(\frac{\operatorname{Arg} x+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|x|} \cdot \sin(\frac{\operatorname{Arg} x+2\cdot\pi\cdot k}{n})) i_{\mathbb{C}_{\rm F}}$ is a complex root of $n$, $x$.

(76) For every element $x$ of $\mathbb{C}_{\rm F}$ and for every complex root $v$ of 1, $x$ holds $v = x$.

(77) For every non empty natural number $n$ and for every complex root $v$ of $n$, $0_{\mathbb{C}_{\rm F}}$ holds $v = 0_{\mathbb{C}_{\rm F}}$.

(78) Let $n$ be a non empty natural number, $x$ be an element of $\mathbb{C}_{\rm F}$, and $v$ be a complex root of $n$, $x$. If $v = 0_{\mathbb{C}_{\rm F}}$, then $x = 0_{\mathbb{C}_{\rm F}}$.

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