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The Complex Numbers

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Summary. We define the set $\mathbb{C}$ of complex numbers as the set of all ordered pairs $z = \langle a, b \rangle$ where a and b are real numbers and where addition and multiplication are defined. We define the real and imaginary parts of z and denote this by $a = \Re(z)$, $b = \Im(z)$. These definitions satisfy all the axioms for a field. $0_{\mathbb{C}} = 0 + 0i$ and $1_{\mathbb{C}} = 1 + 0i$ are identities for addition and multiplication respectively, and there are multiplicative inverses for each non zero element in $\mathbb{C}$. The difference and division of complex numbers are also defined. We do not interpret the set of all real numbers $\mathbb{R}$ as a subset of $\mathbb{C}$. From here on we do not abandon the ordered pair notation for complex numbers. For example: $i^2 = (0 + 1i)^2 = -1 + 0i \neq -1$. We conclude this article by introducing two operations on $\mathbb{C}$ which are not field operations. We define the absolute value of z denoted by $|z|$ and the conjugate of z denoted by z^*.

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The articles [8], [11], [1], [5], [9], [6], [7], [10], [12], [2], [3], and [4] provide the notation and terminology for this paper.

In this paper a, b denote elements of $\mathbb{R}$.

One can prove the following proposition

(2)[1] For all real numbers a, b holds $a^2 + b^2 = 0$ iff $a = 0$ and $b = 0$.

Let us observe that every element of $\mathbb{C}$ is complex.

Let z be a complex number. The functor $\Re(z)$ is defined by:

(Def. 2)[2](i) $\Re(z) = z$ if $z \in \mathbb{R}$,

(ii) there exists a function f from 2 into $\mathbb{R}$ such that $z = f$ and $\Re(z) = f(0)$, otherwise.

The functor $\Im(z)$ is defined as follows:

(Def. 3)(i) $\Im(z) = 0$ if $z \in \mathbb{R}$,

(ii) there exists a function f from 2 into $\mathbb{R}$ such that $z = f$ and $\Im(z) = f(1)$, otherwise.

Let z be a complex number. One can check that $\Re(z)$ is real and $\Im(z)$ is real.

Let z be a complex number. Then $\Re(z)$ is a real number. Then $\Im(z)$ is a real number.

We now state four propositions:

(5)[3] For every function f from 2 into $\mathbb{R}$ there exist a, b such that $f = [0 \longmapsto a, 1 \longmapsto b]$.

[1] The proposition (1) has been removed.
[2] The definition (Def. 1) has been removed.
[3] The propositions (3) and (4) have been removed.

(7)[4] $\Re(a+bi) = a$ and $\Im(a+bi) = b$.

(8) For every complex number z holds $\Re(z) + \Im(z)i = z$.

(9) For all complex numbers z_1, z_2 such that $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$ holds $z_1 = z_2$.

Let z_1, z_2 be complex numbers. Let us observe that $z_1 = z_2$ if and only if:

(Def. 5)[5] $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.

The element $0_{\mathbb{C}}$ of $\mathbb{C}$ is defined as follows:

(Def. 6) $0_{\mathbb{C}} = 0$.

The element $1_{\mathbb{C}}$ of $\mathbb{C}$ is defined as follows:

(Def. 7) $1_{\mathbb{C}} = 1$.

Then i is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 8) $i = 0 + 1i$.

Let us note that $0_{\mathbb{C}}$ is zero.
One can prove the following propositions:

(12)[6] $\Re(0_{\mathbb{C}}) = 0$ and $\Im(0_{\mathbb{C}}) = 0$.

(13) For every complex number z holds $z = 0_{\mathbb{C}}$ iff $\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}} = 0$.

(14) $0 = 0_{\mathbb{C}}$.

(15) $\Re(1_{\mathbb{C}}) = 1$ and $\Im(1_{\mathbb{C}}) = 0$.

(17)[7] $\Re(i) = 0$ and $\Im(i) = 1$.

In the sequel z, z_1, z_2 are elements of $\mathbb{C}$.
Let us consider z_1, z_2. Then $z_1 + z_2$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 9) $z_1 + z_2 = (\Re(z_1) + \Re(z_2)) + (\Im(z_1) + \Im(z_2))i$.

One can prove the following propositions:

(19)[8] For all complex numbers z_1, z_2 holds $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$.

(22)[9] $0_{\mathbb{C}} + z = z$.

Let us consider z_1, z_2. Then $z_1 \cdot z_2$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 10) $z_1 \cdot z_2 = (\Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2)) + (\Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1))i$.

Next we state several propositions:

(24)[10] For all complex numbers z_1, z_2 holds $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1)$.

[4] The proposition (6) has been removed.
[5] The definition (Def. 4) has been removed.
[6] The propositions (10) and (11) have been removed.
[7] The proposition (16) has been removed.
[8] The proposition (18) has been removed.
[9] The propositions (20) and (21) have been removed.
[10] The proposition (23) has been removed.

(28)[11] $0_{\mathbb{C}} \cdot z = 0_{\mathbb{C}}$.

(29) $1_{\mathbb{C}} \cdot z = z$.

(30) If $\Im(z_1) = 0$ and $\Im(z_2) = 0$, then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(31) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$, then $\Re(z_1 \cdot z_2) = -\Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(32) $\Re(z \cdot z) = \Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}$ and $\Im(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im(z))$.

Let us consider z. Then $-z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 11) $-z = -\Re(z) + (-\Im(z))i$.

One can prove the following three propositions:

(34)[12] For every complex number z holds $\Re(-z) = -\Re(z)$ and $\Im(-z) = -\Im(z)$.

(37)[13] $i \cdot i = -1_{\mathbb{C}}$.

(46)[14] $-z = (-1_{\mathbb{C}}) \cdot z$.

Let us consider z_1, z_2. Then $z_1 - z_2$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 12) $z_1 - z_2 = (\Re(z_1) - \Re(z_2)) + (\Im(z_1) - \Im(z_2))i$.

The following two propositions are true:

(48)[15] $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im(z_1 - z_2) = \Im(z_1) - \Im(z_2)$.

(52)[16] $z - 0_{\mathbb{C}} = z$.

Let us consider z. Then z^{-1} is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 13) $z^{-1} = \frac{\Re(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} + \frac{-\Im(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} i$.

We now state several propositions:

(64)[17] For every complex number z holds $\Re(z^{-1}) = \frac{\Re(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$ and $\Im(z^{-1}) = \frac{-\Im(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

(65) If $z \neq 0_{\mathbb{C}}$, then $z \cdot z^{-1} = 1_{\mathbb{C}}$.

(69)[18] If $z_2 \neq 0_{\mathbb{C}}$ and $z_1 \cdot z_2 = 1_{\mathbb{C}}$, then $z_1 = {z_2}^{-1}$.

(71)[19] $(1_{\mathbb{C}})^{-1} = 1_{\mathbb{C}}$.

(72) $(i)^{-1} = -i$.

(79)[20] If $\Re(z) \neq 0$ and $\Im(z) = 0$, then $\Re(z^{-1}) = \Re(z)^{-1}$ and $\Im(z^{-1}) = 0$.

(80) If $\Re(z) = 0$ and $\Im(z) \neq 0$, then $\Re(z^{-1}) = 0$ and $\Im(z^{-1}) = -\Im(z)^{-1}$.

Let us consider z_1, z_2. Then $\frac{z_1}{z_2}$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

[11] The propositions (25)–(27) have been removed.
[12] The proposition (33) has been removed.
[13] The propositions (35) and (36) have been removed.
[14] The propositions (38)–(45) have been removed.
[15] The proposition (47) has been removed.
[16] The propositions (49)–(51) have been removed.
[17] The propositions (53)–(63) have been removed.
[18] The propositions (66)–(68) have been removed.
[19] The proposition (70) has been removed.
[20] The propositions (73)–(78) have been removed.

(Def. 14) $\frac{z_1}{z_2} = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^2+\Im(z_2)^2} + \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^2+\Im(z_2)^2} i$.

Next we state several propositions:

(82)[21] $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^2+\Im(z_2)^2}$ and $\Im(\frac{z_1}{z_2}) = \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^2+\Im(z_2)^2}$.

(84)[22] If $z \neq 0_{\mathbb{C}}$, then $z^{-1} = \frac{1_{\mathbb{C}}}{z}$.

(85) $\frac{z}{1_{\mathbb{C}}} = z$.

(86) If $z \neq 0_{\mathbb{C}}$, then $\frac{z}{z} = 1_{\mathbb{C}}$.

(87) $\frac{0_{\mathbb{C}}}{z} = 0_{\mathbb{C}}$.

(91)[23] If $z_2 \neq 0_{\mathbb{C}}$ and $\frac{z_1}{z_2} = 1_{\mathbb{C}}$, then $z_1 = z_2$.

(109)[24] If $\Im(z_1) = 0$ and $\Im(z_2) = 0$ and $\Re(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)}{\Re(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

(110) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$ and $\Im(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Im(z_1)}{\Im(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

Let z be a complex number. The functor $\overline{z}$ yielding a complex number is defined by:

(Def. 15) $\overline{z} = \Re(z) + (-\Im(z))i$.

Let us notice that the functor $\overline{z}$ is involutive.

Let z be a complex number. Then $\overline{z}$ is an element of $\mathbb{C}$.

Next we state a number of propositions:

(112)[25] For every complex number z holds $\Re(\overline{z}) = \Re(z)$ and $\Im(\overline{z}) = -\Im(z)$.

(113) $\overline{0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(114) If $\overline{z} = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(115) $\overline{1_{\mathbb{C}}} = 1_{\mathbb{C}}$.

(116) $\overline{i} = -i$.

(118)[26] $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(119) $\overline{-z} = -\overline{z}$.

(120) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(121) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

(122) $\overline{z^{-1}} = \overline{z}^{-1}$.

(123) $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$.

(124) If $\Im(z) = 0$, then $\overline{z} = z$.

(125) If $\Re(z) = 0$, then $\overline{z} = -z$.

(126) $\Re(z \cdot \overline{z}) = \Re(z)^2 + \Im(z)^2$ and $\Im(z \cdot \overline{z}) = 0$.

(127) $\Re(z + \overline{z}) = 2 \cdot \Re(z)$ and $\Im(z + \overline{z}) = 0$.

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[21] The proposition (81) has been removed.
[22] The proposition (83) has been removed.
[23] The propositions (88)–(90) have been removed.
[24] The propositions (92)–(108) have been removed.
[25] The proposition (111) has been removed.
[26] The proposition (117) has been removed.

(128) $\Re(z-\overline{z})=0$ and $\Im(z-\overline{z})=2\cdot\Im(z)$.

Let z be a complex number. The functor $|z|$ is defined as follows:

(Def. 16) $|z|=\sqrt{\Re(z)^{2}+\Im(z)^{2}}$.

Let z be a complex number. Note that $|z|$ is real.
Let z be a complex number. Then $|z|$ is a real number.
We now state several propositions:

(130)[27] $|0_{\mathbb{C}}|=0$.

(131) For every complex number z such that $|z|=0$ holds $z=0_{\mathbb{C}}$.

(132) For every complex number z holds $0\leq|z|$.

(133) For every complex number z holds $z\neq0_{\mathbb{C}}$ iff $0<|z|$.

(134) $|1_{\mathbb{C}}|=1$.

(135) $|i|=1$.

(136) For every complex number z such that $\Im(z)=0$ holds $|z|=|\Re(z)|$.

(137) For every complex number z such that $\Re(z)=0$ holds $|z|=|\Im(z)|$.

(138) For every complex number z holds $|-z|=|z|$.

In the sequel z is a complex number.
We now state a number of propositions:

(139) $|\overline{z}|=|z|$.

(140) $\Re(z)\leq|z|$.

(141) $\Im(z)\leq|z|$.

(142) For all complex numbers z_1, z_2 holds $|z_1+z_2|\leq|z_1|+|z_2|$.

(143) $|z_1-z_2|\leq|z_1|+|z_2|$.

(144) $|z_1|-|z_2|\leq|z_1+z_2|$.

(145) $|z_1|-|z_2|\leq|z_1-z_2|$.

(146) $|z_1-z_2|=|z_2-z_1|$.

(147) $|z_1-z_2|=0$ iff $z_1=z_2$.

(148) $z_1\neq z_2$ iff $0<|z_1-z_2|$.

(149) $|z_1-z_2|\leq|z_1-z|+|z-z_2|$.

(150) $||z_1|-|z_2||\leq|z_1-z_2|$.

(151) For all complex numbers z_1, z_2 holds $|z_1\cdot z_2|=|z_1|\cdot|z_2|$.

(152) If $z\neq0_{\mathbb{C}}$, then $|z^{-1}|=|z|^{-1}$.

(153) If $z_2\neq0_{\mathbb{C}}$, then $\frac{|z_1|}{|z_2|}=|\frac{z_1}{z_2}|$.

(154) $|z\cdot z|=\Re(z)^{2}+\Im(z)^{2}$.

(155) $|z\cdot z|=|z\cdot\overline{z}|$.

[27] The proposition (129) has been removed.

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