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Coherent Space

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Summary. Coherent Space, web of coherent space and two categories: category of coherent spaces and category of tolerances on same fixed set.

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The articles [9], [6], [11], [12], [13], [10], [2], [5], [1], [3], [8], [7], and [4] provide the notation and terminology for this paper.

1. Coherent Space and Web of Coherent Space

In this paper x, y, a, b, X, A denote sets.

Let I_1 be a set. We say that I_1 is binary complete if and only if:

(Def. 2)[1] For every A such that $A \subseteq I_1$ and for all a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in I_1$ holds $\bigcup A \in I_1$.

Let us note that there exists a set which is subset-closed, binary complete, and non empty.

A coherent space is a subset-closed binary complete non empty set.

In the sequel C, D denote coherent spaces.

One can prove the following propositions:

(1) $\emptyset \in C$.

(2) 2^X is a coherent space.

(3) $\{\emptyset\}$ is a coherent space.

(4) If $x \in \bigcup C$, then $\{x\} \in C$.

Let C be a coherent space. The functor $\mathrm{Web}(C)$ yields a tolerance of $\bigcup C$ and is defined by:

(Def. 3) For all x, y holds $\langle x, y\rangle \in \mathrm{Web}(C)$ iff there exists X such that $X \in C$ and $x \in X$ and $y \in X$.

In the sequel T denotes a tolerance of $\bigcup C$.

The following propositions are true:

(5) $T = \mathrm{Web}(C)$ iff for all x, y holds $\langle x, y\rangle \in T$ iff $\{x,y\} \in C$.

(6) $a \in C$ iff for all x, y such that $x \in a$ and $y \in a$ holds $\{x,y\} \in C$.

[1] The definition (Def. 1) has been removed.

(7) $a \in C$ iff for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in \mathrm{Web}(C)$.

(8) If for all x, y such that $x \in a$ and $y \in a$ holds $\{x, y\} \in C$, then $a \subseteq \bigcup C$.

(9) If $\mathrm{Web}(C) = \mathrm{Web}(D)$, then $C = D$.

(10) If $\bigcup C \in C$, then $C = 2^{\bigcup C}$.

(11) If $C = 2^{\bigcup C}$, then $\mathrm{Web}(C) = \nabla_{\bigcup C}$.

Let X be a set and let E be a tolerance of X. The functor $\mathrm{CohSp}(E)$ yields a coherent space and is defined by:

(Def. 4) For every a holds $a \in \mathrm{CohSp}(E)$ iff for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in E$.

In the sequel E is a tolerance of X.

We now state four propositions:

(12) $\mathrm{Web}(\mathrm{CohSp}(E)) = E$.

(13) $\mathrm{CohSp}(\mathrm{Web}(C)) = C$.

(14) $a \in \mathrm{CohSp}(E)$ iff a is a set of mutually elements w.r.t. E.

(15) $\mathrm{CohSp}(E) = \mathrm{TolSets}\, E$.

2. Category of Coherent Spaces

Let us consider X. The functor $\mathrm{CSp}(X)$ yields a set and is defined by:

(Def. 5) $\mathrm{CSp}(X) = \{x; x$ ranges over subsets of 2^X: x is a coherent space$\}$.

Let us consider X. One can verify that $\mathrm{CSp}(X)$ is non empty.

Let X be a set. Note that every element of $\mathrm{CSp}(X)$ is subset-closed, binary complete, and non empty.

In the sequel C, C_1, C_2 are elements of $\mathrm{CSp}(X)$.

One can prove the following proposition

(16) If $\{x, y\} \in C$, then $x \in \bigcup C$ and $y \in \bigcup C$.

Let us consider X. The functor $\mathrm{Funcs_C} X$ yields a set and is defined by:

(Def. 7)[2] $\mathrm{Funcs_C} X = \bigcup\{(\bigcup y)^{\bigcup x} : x$ ranges over elements of $\mathrm{CSp}(X)$, y ranges over elements of $\mathrm{CSp}(X)\}$.

Let us consider X. One can verify that $\mathrm{Funcs_C} X$ is non empty and functional.

In the sequel g denotes an element of $\mathrm{Funcs_C} X$.

One can prove the following proposition

(17) $x \in \mathrm{Funcs_C} X$ iff there exist C_1, C_2 such that if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and x is a function from $\bigcup C_1$ into $\bigcup C_2$.

Let us consider X. The functor $\mathrm{Maps_C} X$ yields a set and is defined by the condition (Def. 8).

(Def. 8) $\mathrm{Maps_C} X = \{\langle\langle C, C_3\rangle, f\rangle; C$ ranges over elements of $\mathrm{CSp}(X)$, C_3 ranges over elements of $\mathrm{CSp}(X)$, f ranges over elements of $\mathrm{Funcs_C} X$: $(\bigcup C_3 = \emptyset \Rightarrow \bigcup C = \emptyset) \wedge f$ is a function from $\bigcup C$ into $\bigcup C_3 \wedge \bigwedge_{x,y} (\{x, y\} \in C \Rightarrow \{f(x), f(y)\} \in C_3)\}$.

Let us consider X. One can check that $\mathrm{Maps_C} X$ is non empty.

In the sequel l, l_1, l_2, l_3 denote elements of $\mathrm{Maps_C} X$.

One can prove the following two propositions:

[2] The definition (Def. 6) has been removed.

(18) There exist g, C_1, C_2 such that

(i) $l = \langle\langle C_1, C_2\rangle, g\rangle$,

(ii) if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$,

(iii) g is a function from $\bigcup C_1$ into $\bigcup C_2$, and

(iv) for all x, y such that $\{x, y\} \in C_1$ holds $\{g(x), g(y)\} \in C_2$.

(19) Let f be a function from $\bigcup C_1$ into $\bigcup C_2$. Suppose if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and for all x, y such that $\{x, y\} \in C_1$ holds $\{f(x), f(y)\} \in C_2$. Then $\langle\langle C_1, C_2\rangle, f\rangle \in \mathrm{Maps}_{\mathrm{C}} X$.

Let X be a set and let l be an element of $\mathrm{Maps}_{\mathrm{C}} X$. Note that $l_{\mathbf{2}}$ is function-like and relation-like. Let us consider X, l. The functor $\mathrm{dom}\, l$ yields an element of $\mathrm{CSp}(X)$ and is defined as follows:

(Def. 10)[3] $\mathrm{dom}\, l = (l_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, l$ yielding an element of $\mathrm{CSp}(X)$ is defined by:

(Def. 11) $\mathrm{cod}\, l = (l_{\mathbf{1}})_{\mathbf{2}}$.

One can prove the following proposition

(20) $l = \langle\langle \mathrm{dom}\, l, \mathrm{cod}\, l\rangle, l_{\mathbf{2}}\rangle$.

Let us consider X, C. The functor $\mathrm{id}(C)$ yielding an element of $\mathrm{Maps}_{\mathrm{C}} X$ is defined as follows:

(Def. 12) $\mathrm{id}(C) = \langle\langle C, C\rangle, \mathrm{id}_{\bigcup C}\rangle$.

The following proposition is true

(21) $\bigcup \mathrm{cod}\, l \neq \emptyset$ or $\bigcup \mathrm{dom}\, l = \emptyset$ but $l_{\mathbf{2}}$ is a function from $\bigcup \mathrm{dom}\, l$ into $\bigcup \mathrm{cod}\, l$ but for all x, y such that $\{x, y\} \in \mathrm{dom}\, l$ holds $\{l_{\mathbf{2}}(x), l_{\mathbf{2}}(y)\} \in \mathrm{cod}\, l$.

Let us consider X, l_1, l_2. Let us assume that $\mathrm{cod}\, l_1 = \mathrm{dom}\, l_2$. The functor $l_2 \cdot l_1$ yields an element of $\mathrm{Maps}_{\mathrm{C}} X$ and is defined by:

(Def. 13) $l_2 \cdot l_1 = \langle\langle \mathrm{dom}\, l_1, \mathrm{cod}\, l_2\rangle, (l_2)_{\mathbf{2}} \cdot (l_1)_{\mathbf{2}}\rangle$.

One can prove the following propositions:

(22) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$, then $(l_2 \cdot l_1)_{\mathbf{2}} = (l_2)_{\mathbf{2}} \cdot (l_1)_{\mathbf{2}}$ and $\mathrm{dom}(l_2 \cdot l_1) = \mathrm{dom}\, l_1$ and $\mathrm{cod}(l_2 \cdot l_1) = \mathrm{cod}\, l_2$.

(23) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and $\mathrm{dom}\, l_3 = \mathrm{cod}\, l_2$, then $l_3 \cdot (l_2 \cdot l_1) = (l_3 \cdot l_2) \cdot l_1$.

(24) $(\mathrm{id}(C))_{\mathbf{2}} = \mathrm{id}_{\bigcup C}$ and $\mathrm{dom}\,\mathrm{id}(C) = C$ and $\mathrm{cod}\,\mathrm{id}(C) = C$.

(25) $l \cdot \mathrm{id}(\mathrm{dom}\, l) = l$ and $\mathrm{id}(\mathrm{cod}\, l) \cdot l = l$.

Let us consider X. The functor $\mathrm{Dom}_{\mathrm{CSp}} X$ yields a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ and is defined by:

(Def. 14) For every l holds $(\mathrm{Dom}_{\mathrm{CSp}} X)(l) = \mathrm{dom}\, l$.

The functor $\mathrm{Cod}_{\mathrm{CSp}} X$ yields a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ and is defined by:

(Def. 15) For every l holds $(\mathrm{Cod}_{\mathrm{CSp}} X)(l) = \mathrm{cod}\, l$.

The functor $\cdot_{\mathrm{CSp}} X$ yields a partial function from $[: \mathrm{Maps}_{\mathrm{C}} X, \mathrm{Maps}_{\mathrm{C}} X :]$ to $\mathrm{Maps}_{\mathrm{C}} X$ and is defined as follows:

(Def. 16) For all l_2, l_1 holds $\langle l_2, l_1\rangle \in \mathrm{dom} \cdot_{\mathrm{CSp}} X$ iff $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and for all l_2, l_1 such that $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ holds $(\cdot_{\mathrm{CSp}} X)(\langle l_2, l_1\rangle) = l_2 \cdot l_1$.

[3] The definition (Def. 9) has been removed.

The functor $\mathrm{Id}_{\mathrm{CSp}}X$ yields a function from $\mathrm{CSp}(X)$ into $\mathrm{Maps}_{\mathrm{C}}X$ and is defined by:

(Def. 17) For every C holds $(\mathrm{Id}_{\mathrm{CSp}}X)(C) = \mathrm{id}(C)$.

We now state the proposition

(26) $\langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}}X, \mathrm{Dom}_{\mathrm{CSp}}X, \mathrm{Cod}_{\mathrm{CSp}}X, \cdot_{\mathrm{CSp}}X, \mathrm{Id}_{\mathrm{CSp}}X\rangle$ is a category.

Let us consider X. The X-coherent space category yielding a category is defined by:

(Def. 18) The X-coherent space category $= \langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}}X, \mathrm{Dom}_{\mathrm{CSp}}X, \mathrm{Cod}_{\mathrm{CSp}}X, \cdot_{\mathrm{CSp}}X, \mathrm{Id}_{\mathrm{CSp}}X\rangle$.

3. Category of Tolerances

Let X be a set. The tolerances on X constitute a set defined as follows:

(Def. 19) $x \in$ the tolerances on X iff x is a tolerance of X.

Let X be a set. Note that the tolerances on X is non empty.
Let X be a set. The tolerances on subsets of X constitute a set defined as follows:

(Def. 20) The tolerances on subsets of $X = \bigcup\{$the tolerances on Y: Y ranges over subsets of $X\}$.

Let X be a set. One can check that the tolerances on subsets of X is non empty.
One can prove the following propositions:

(27) $x \in$ the tolerances on subsets of X iff there exists A such that $A \subseteq X$ and x is a tolerance of A.

(28) $\nabla_a \in$ the tolerances on a.

(30)[4] $\emptyset \in$ the tolerances on subsets of X.

(31) If $a \subseteq X$, then $\nabla_a \in$ the tolerances on subsets of X.

(32) If $a \subseteq X$, then $\mathrm{id}_a \in$ the tolerances on subsets of X.

(33) $\nabla_X \in$ the tolerances on subsets of X.

(34) $\mathrm{id}_X \in$ the tolerances on subsets of X.

Let us consider X. The functor $\mathrm{TOL}(X)$ yields a set and is defined by:

(Def. 21) $\mathrm{TOL}(X) = \{\langle t, Y\rangle; t$ ranges over elements of the tolerances on subsets of X, Y ranges over elements of 2^X: t is a tolerance of $Y\}$.

Let us consider X. One can verify that $\mathrm{TOL}(X)$ is non empty.
In the sequel T, T_1, T_2 denote elements of $\mathrm{TOL}(X)$.
One can prove the following propositions:

(35) $\langle\emptyset, \emptyset\rangle \in \mathrm{TOL}(X)$.

(36) If $a \subseteq X$, then $\langle \mathrm{id}_a, a\rangle \in \mathrm{TOL}(X)$.

(37) If $a \subseteq X$, then $\langle \nabla_a, a\rangle \in \mathrm{TOL}(X)$.

(38) $\langle \mathrm{id}_X, X\rangle \in \mathrm{TOL}(X)$.

(39) $\langle \nabla_X, X\rangle \in \mathrm{TOL}(X)$.

Let us consider X, T. Then $T_{\mathbf{2}}$ is an element of 2^X. Then $T_{\mathbf{1}}$ is a tolerance of $T_{\mathbf{2}}$.
Let us consider X. The functor $\mathrm{Funcs}_{\mathrm{T}}X$ yields a set and is defined by:

[4] The proposition (29) has been removed.

(Def. 22) $\mathrm{Funcs_T}X = \bigcup\{((T_3)_{\mathbf{2}})^{T_2} : T$ ranges over elements of $\mathrm{TOL}(X)$, T_3 ranges over elements of $\mathrm{TOL}(X)\}$.

Let us consider X. Observe that $\mathrm{Funcs_T}X$ is non empty and functional.
In the sequel f is an element of $\mathrm{Funcs_T}X$.
One can prove the following proposition

(40) $x \in \mathrm{Funcs_T}X$ iff there exist T_1, T_2 such that if $(T_2)_{\mathbf{2}} = \emptyset$, then $(T_1)_{\mathbf{2}} = \emptyset$ and x is a function from $(T_1)_{\mathbf{2}}$ into $(T_2)_{\mathbf{2}}$.

Let us consider X. The functor $\mathrm{Maps_T}X$ yielding a set is defined by the condition (Def. 23).

(Def. 23) $\mathrm{Maps_T}X = \{\langle\langle T, T_3\rangle, f\rangle; T$ ranges over elements of $\mathrm{TOL}(X)$, T_3 ranges over elements of $\mathrm{TOL}(X)$, f ranges over elements of $\mathrm{Funcs_T}X : ((T_3)_{\mathbf{2}} = \emptyset \Rightarrow T_{\mathbf{2}} = \emptyset) \wedge f$ is a function from $T_{\mathbf{2}}$ into $(T_3)_{\mathbf{2}} \wedge \bigwedge_{x,y} (\langle x, y\rangle \in T_{\mathbf{1}} \Rightarrow \langle f(x), f(y)\rangle \in (T_3)_{\mathbf{1}})\}$.

Let us consider X. Note that $\mathrm{Maps_T}X$ is non empty.
In the sequel m, m_1, m_2, m_3 are elements of $\mathrm{Maps_T}X$.
The following propositions are true:

(41) There exist f, T_1, T_2 such that

(i) $m = \langle\langle T_1, T_2\rangle, f\rangle$,

(ii) if $(T_2)_{\mathbf{2}} = \emptyset$, then $(T_1)_{\mathbf{2}} = \emptyset$,

(iii) f is a function from $(T_1)_{\mathbf{2}}$ into $(T_2)_{\mathbf{2}}$, and

(iv) for all x, y such that $\langle x, y\rangle \in (T_1)_{\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in (T_2)_{\mathbf{1}}$.

(42) Let f be a function from $(T_1)_{\mathbf{2}}$ into $(T_2)_{\mathbf{2}}$. Suppose if $(T_2)_{\mathbf{2}} = \emptyset$, then $(T_1)_{\mathbf{2}} = \emptyset$ and for all x, y such that $\langle x, y\rangle \in (T_1)_{\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in (T_2)_{\mathbf{1}}$. Then $\langle\langle T_1, T_2\rangle, f\rangle \in \mathrm{Maps_T}X$.

Let X be a set and let m be an element of $\mathrm{Maps_T}X$. One can check that $m_{\mathbf{2}}$ is function-like and relation-like.

Let us consider X, m. The functor $\mathrm{dom}\, m$ yields an element of $\mathrm{TOL}(X)$ and is defined by:

(Def. 25)[5] $\mathrm{dom}\, m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, m$ yielding an element of $\mathrm{TOL}(X)$ is defined as follows:

(Def. 26) $\mathrm{cod}\, m = (m_{\mathbf{1}})_{\mathbf{2}}$.

Next we state the proposition

(43) $m = \langle\langle \mathrm{dom}\, m, \mathrm{cod}\, m\rangle, m_{\mathbf{2}}\rangle$.

Let us consider X, T. The functor $\mathrm{id}(T)$ yields an element of $\mathrm{Maps_T}X$ and is defined as follows:

(Def. 27) $\mathrm{id}(T) = \langle\langle T, T\rangle, \mathrm{id}_{T_{\mathbf{2}}}\rangle$.

Next we state the proposition

(44) $(\mathrm{cod}\, m)_{\mathbf{2}} \neq \emptyset$ or $(\mathrm{dom}\, m)_{\mathbf{2}} = \emptyset$ but $m_{\mathbf{2}}$ is a function from $(\mathrm{dom}\, m)_{\mathbf{2}}$ into $(\mathrm{cod}\, m)_{\mathbf{2}}$ but for all x, y such that $\langle x, y\rangle \in (\mathrm{dom}\, m)_{\mathbf{1}}$ holds $\langle m_{\mathbf{2}}(x), m_{\mathbf{2}}(y)\rangle \in (\mathrm{cod}\, m)_{\mathbf{1}}$.

Let us consider X, m_1, m_2. Let us assume that $\mathrm{cod}\, m_1 = \mathrm{dom}\, m_2$. The functor $m_2 \cdot m_1$ yielding an element of $\mathrm{Maps_T}X$ is defined by:

(Def. 28) $m_2 \cdot m_1 = \langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_2\rangle, (m_2)_{\mathbf{2}} \cdot (m_1)_{\mathbf{2}}\rangle$.

The following four propositions are true:

[5] The definition (Def. 24) has been removed.

(45) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$, then $(m_2 \cdot m_1)_{\mathbf{2}} = (m_2)_{\mathbf{2}} \cdot (m_1)_{\mathbf{2}}$ and $\mathrm{dom}(m_2 \cdot m_1) = \mathrm{dom}\, m_1$ and $\mathrm{cod}(m_2 \cdot m_1) = \mathrm{cod}\, m_2$.

(46) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and $\mathrm{dom}\, m_3 = \mathrm{cod}\, m_2$, then $m_3 \cdot (m_2 \cdot m_1) = (m_3 \cdot m_2) \cdot m_1$.

(47) $(\mathrm{id}(T))_{\mathbf{2}} = \mathrm{id}_{T_{\mathbf{2}}}$ and $\mathrm{dom}\,\mathrm{id}(T) = T$ and $\mathrm{cod}\,\mathrm{id}(T) = T$.

(48) $m \cdot \mathrm{id}(\mathrm{dom}\, m) = m$ and $\mathrm{id}(\mathrm{cod}\, m) \cdot m = m$.

Let us consider X. The functor Dom_X yields a function from $\mathrm{Maps}_{\mathrm{T}} X$ into $\mathrm{TOL}(X)$ and is defined as follows:

(Def. 29) For every m holds $\mathrm{Dom}_X(m) = \mathrm{dom}\, m$.

The functor Cod_X yields a function from $\mathrm{Maps}_{\mathrm{T}} X$ into $\mathrm{TOL}(X)$ and is defined as follows:

(Def. 30) For every m holds $\mathrm{Cod}_X(m) = \mathrm{cod}\, m$.

The functor $\cdot_X$ yields a partial function from $[: \mathrm{Maps}_{\mathrm{T}} X, \mathrm{Maps}_{\mathrm{T}} X :]$ to $\mathrm{Maps}_{\mathrm{T}} X$ and is defined by:

(Def. 31) For all m_2, m_1 holds $\langle m_2, m_1 \rangle \in \mathrm{dom}(\cdot_X)$ iff $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and for all m_2, m_1 such that $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ holds $\cdot_X(\langle m_2, m_1 \rangle) = m_2 \cdot m_1$.

The functor Id_X yielding a function from $\mathrm{TOL}(X)$ into $\mathrm{Maps}_{\mathrm{T}} X$ is defined by:

(Def. 32) For every T holds $\mathrm{Id}_X(T) = \mathrm{id}(T)$.

Next we state the proposition

(49) $\langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}} X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X \rangle$ is a category.

Let us consider X. The X-tolerance category is a category and is defined by:

(Def. 33) The X-tolerance category $= \langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}} X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X \rangle$.

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