JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Tarski's Classes and Ranks

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the article the Tarski's classes (non-empty families of sets satisfying Tarski's axiom A given in [7]) and the rank sets are introduced and some of their properties are shown. The transitive closure and the rank of a set is given here too.

MML Identifier: CLASSES1.

WWW: http://mizar.org/JFM/Vol2/classes1.html

The articles [7], [6], [9], [10], [5], [8], [2], [3], [4], and [1] provide the notation and terminology for this paper.

We adopt the following rules: W, X, Y, Z are sets, f is a function, and x, y are sets.

Let B be a set. We say that B is subset-closed if and only if:

(Def. 1) For all X, Y such that $X \in B$ and $Y \subseteq X$ holds $Y \in B$.

Let B be a set. We say that B is a Tarski class if and only if:

(Def. 2) B is subset-closed and for every X such that $X \in B$ holds $2^X \in B$ and for every X such that $X \subseteq B$ holds $X \approx B$ or $X \in B$.

We introduce B is a Tarski class as a synonym of B is a Tarski class.

Let A, B be sets. We say that B is a Tarski class of A if and only if:

(Def. 3) $A \in B$ and B is a Tarski class.

Let A be a set. The functor $\mathbf{T}(A)$ yielding a set is defined as follows:

(Def. 4) $\mathbf{T}(A)$ is a Tarski class of A and for every set D such that D is a Tarski class of A holds $\mathbf{T}(A) \subseteq D$.

Let A be a set. Note that $\mathbf{T}(A)$ is non empty.

The following propositions are true:

(2)[1] W is a Tarski class if and only if the following conditions are satisfied:

(i) W is subset-closed,

(ii) for every X such that $X \in W$ holds $2^X \in W$, and

(iii) for every X such that $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$ holds $X \in W$.

(5)[2] $X \in \mathbf{T}(X)$.

(6) If $Y \in \mathbf{T}(X)$ and $Z \subseteq Y$, then $Z \in \mathbf{T}(X)$.

[1] The proposition (1) has been removed.
[2] The propositions (3) and (4) have been removed.

(7) If $Y \in \mathbf{T}(X)$, then $2^Y \in \mathbf{T}(X)$.

(8) If $Y \subseteq \mathbf{T}(X)$, then $Y \approx \mathbf{T}(X)$ or $Y \in \mathbf{T}(X)$.

(9) If $Y \subseteq \mathbf{T}(X)$ and $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$, then $Y \in \mathbf{T}(X)$.

We use the following convention: u, v are elements of $\mathbf{T}(X)$, A, B, C are ordinal numbers, and L is a transfinite sequence.

Let us consider X, A. The functor $\mathbf{T}_A(X)$ is defined by the condition (Def. 5).

(Def. 5) There exists L such that

(i) $\mathbf{T}_A(X) = \mathrm{last}\, L$,

(ii) $\mathrm{dom}\, L = \mathrm{succ}\, A$,

(iii) $L(\emptyset) = \{X\}$,

(iv) for every C such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ holds $L(\mathrm{succ}\, C) = \{u : \bigvee_v \, (v \in L(C) \wedge u \subseteq v)\} \cup \{2^v : v \in L(C)\} \cup 2^{L(C)} \cap \mathbf{T}(X)$, and

(v) for every C such that $C \in \mathrm{succ}\, A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \bigcup \mathrm{rng}(L{\restriction}C) \cap \mathbf{T}(X)$.

Let us consider X, A. Then $\mathbf{T}_A(X)$ is a subset of $\mathbf{T}(X)$.

One can prove the following propositions:

(10) $\mathbf{T}_\emptyset(X) = \{X\}$.

(11) $\mathbf{T}_{\mathrm{succ}\, A}(X) = \{u : \bigvee_v \, (v \in \mathbf{T}_A(X) \wedge u \subseteq v)\} \cup \{2^v : v \in \mathbf{T}_A(X)\} \cup 2^{\mathbf{T}_A(X)} \cap \mathbf{T}(X)$.

(12) If $A \neq \emptyset$ and A is a limit ordinal number, then $\mathbf{T}_A(X) = \{u : \bigvee_B \, (B \in A \wedge u \in \mathbf{T}_B(X))\}$.

(13) $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$ iff $Y \subseteq \mathbf{T}_A(X)$ and $Y \in \mathbf{T}(X)$ or there exists Z such that $Z \in \mathbf{T}_A(X)$ but $Y \subseteq Z$ or $Y = 2^Z$.

(14) If $Y \subseteq Z$ and $Z \in \mathbf{T}_A(X)$, then $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(15) If $Y \in \mathbf{T}_A(X)$, then $2^Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(16) If $A \neq \emptyset$ and A is a limit ordinal number, then $x \in \mathbf{T}_A(X)$ iff there exists B such that $B \in A$ and $x \in \mathbf{T}_B(X)$.

(17) If $A \neq \emptyset$ and A is a limit ordinal number and $Y \in \mathbf{T}_A(X)$ and $Z \subseteq Y$ or $Z = 2^Y$, then $Z \in \mathbf{T}_A(X)$.

(18) $\mathbf{T}_A(X) \subseteq \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(19) If $A \subseteq B$, then $\mathbf{T}_A(X) \subseteq \mathbf{T}_B(X)$.

(20) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(21) If $\mathbf{T}_A(X) = \mathbf{T}_{\mathrm{succ}\, A}(X)$, then $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(22) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(23) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$ and for every B such that $B \in A$ holds $\mathbf{T}_B(X) \neq \mathbf{T}(X)$.

(24) If $Y \neq X$ and $Y \in \mathbf{T}(X)$, then there exists A such that $Y \notin \mathbf{T}_A(X)$ and $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(25) If X is transitive, then for every A such that $A \neq \emptyset$ holds $\mathbf{T}_A(X)$ is transitive.

(26) $\mathbf{T}_\emptyset(X) \in \mathbf{T}_{\mathbf{1}}(X)$ and $\mathbf{T}_\emptyset(X) \neq \mathbf{T}_{\mathbf{1}}(X)$.

(27) If X is transitive, then $\mathbf{T}(X)$ is transitive.

(28) If $Y \in \mathbf{T}(X)$, then $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$.

(29) If $Y \in \mathbf{T}(X)$, then $Y \not\approx \mathbf{T}(X)$.

(30) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\{x\} \in \mathbf{T}(X)$ and $\{x, y\} \in \mathbf{T}(X)$.

(31) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\langle x, y\rangle \in \mathbf{T}(X)$.

(32) If $Y \subseteq \mathbf{T}(X)$ and $Z \subseteq \mathbf{T}(X)$, then $[: Y, Z :] \subseteq \mathbf{T}(X)$.

Let us consider A. The functor $\mathbf{R}_A$ is defined by the condition (Def. 6).

(Def. 6) There exists L such that

(i) $\mathbf{R}_A = \mathrm{last}\, L$,

(ii) $\mathrm{dom}\, L = \mathrm{succ}\, A$,

(iii) $L(\emptyset) = \emptyset$,

(iv) for every C such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ holds $L(\mathrm{succ}\, C) = 2^{L(C)}$, and

(v) for every C such that $C \in \mathrm{succ}\, A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \bigcup \mathrm{rng}(L{\restriction}C)$.

Next we state a number of propositions:

(33) $\mathbf{R}_\emptyset = \emptyset$.

(34) $\mathbf{R}_{\mathrm{succ}\, A} = 2^{\mathbf{R}_A}$.

(35) If $A \neq \emptyset$ and A is a limit ordinal number, then for every x holds $x \in \mathbf{R}_A$ iff there exists B such that $B \in A$ and $x \in \mathbf{R}_B$.

(36) $X \subseteq \mathbf{R}_A$ iff $X \in \mathbf{R}_{\mathrm{succ}\, A}$.

(37) $\mathbf{R}_A$ is transitive.

(38) If $X \in \mathbf{R}_A$, then $X \subseteq \mathbf{R}_A$.

(39) $\mathbf{R}_A \subseteq \mathbf{R}_{\mathrm{succ}\, A}$.

(40) $\bigcup(\mathbf{R}_A) \subseteq \mathbf{R}_A$.

(41) If $X \in \mathbf{R}_A$, then $\bigcup X \in \mathbf{R}_A$.

(42) $A \in B$ iff $\mathbf{R}_A \in \mathbf{R}_B$.

(43) $A \subseteq B$ iff $\mathbf{R}_A \subseteq \mathbf{R}_B$.

(44) $A \subseteq \mathbf{R}_A$.

(45) For all A, X such that $X \in \mathbf{R}_A$ holds $X \not\approx \mathbf{R}_A$ and $\overline{\overline{X}} < \overline{\overline{\mathbf{R}_A}}$.

(46) $X \subseteq \mathbf{R}_A$ iff $2^X \subseteq \mathbf{R}_{\mathrm{succ}\, A}$.

(47) If $X \subseteq Y$ and $Y \in \mathbf{R}_A$, then $X \in \mathbf{R}_A$.

(48) $X \in \mathbf{R}_A$ iff $2^X \in \mathbf{R}_{\mathrm{succ}\, A}$.

(49) $x \in \mathbf{R}_A$ iff $\{x\} \in \mathbf{R}_{\mathrm{succ}\, A}$.

(50) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ iff $\{x, y\} \in \mathbf{R}_{\mathrm{succ}\, A}$.

(51) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ iff $\langle x, y\rangle \in \mathbf{R}_{\mathrm{succ}\,\mathrm{succ}\, A}$.

(52) If X is transitive and $\mathbf{R}_A \cap \mathbf{T}(X) = \mathbf{R}_{\mathrm{succ}\, A} \cap \mathbf{T}(X)$, then $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(53) If X is transitive, then there exists A such that $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(54) If X is transitive, then $\bigcup X \subseteq X$.

(55) If X is transitive and Y is transitive, then $X \cup Y$ is transitive.

(56) If X is transitive and Y is transitive, then $X \cap Y$ is transitive.

In the sequel k, n denote natural numbers.
Let us consider X. The functor $X^{*\in}$ yielding a set is defined by:

(Def. 7) $x \in X^{*\in}$ iff there exist f, n such that $x \in f(n)$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = X$ and for every k holds $f(k+1) = \bigcup f(k)$.

We now state a number of propositions:

(58)[3] $X^{*\in}$ is transitive.

(59) $X \subseteq X^{*\in}$.

(60) If $X \subseteq Y$ and Y is transitive, then $X^{*\in} \subseteq Y$.

(61) If for every Z such that $X \subseteq Z$ and Z is transitive holds $Y \subseteq Z$ and $X \subseteq Y$ and Y is transitive, then $X^{*\in} = Y$.

(62) If X is transitive, then $X^{*\in} = X$.

(63) $\emptyset^{*\in} = \emptyset$.

(64) $A^{*\in} = A$.

(65) If $X \subseteq Y$, then $X^{*\in} \subseteq Y^{*\in}$.

(66) $(X^{*\in})^{*\in} = X^{*\in}$.

(67) $(X \cup Y)^{*\in} = X^{*\in} \cup Y^{*\in}$.

(68) $(X \cap Y)^{*\in} \subseteq X^{*\in} \cap Y^{*\in}$.

(69) There exists A such that $X \subseteq \mathbf{R}_A$.

Let us consider X. The functor $\operatorname{rk}(X)$ yields an ordinal number and is defined as follows:

(Def. 8) $X \subseteq \mathbf{R}_{\operatorname{rk}(X)}$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $\operatorname{rk}(X) \subseteq B$.

Next we state a number of propositions:

(71)[4] $\operatorname{rk}(2^X) = \operatorname{succ}\operatorname{rk}(X)$.

(72) $\operatorname{rk}(\mathbf{R}_A) = A$.

(73) $X \subseteq \mathbf{R}_A$ iff $\operatorname{rk}(X) \subseteq A$.

(74) $X \in \mathbf{R}_A$ iff $\operatorname{rk}(X) \in A$.

(75) If $X \subseteq Y$, then $\operatorname{rk}(X) \subseteq \operatorname{rk}(Y)$.

(76) If $X \in Y$, then $\operatorname{rk}(X) \in \operatorname{rk}(Y)$.

(77) $\operatorname{rk}(X) \subseteq A$ iff for every Y such that $Y \in X$ holds $\operatorname{rk}(Y) \in A$.

(78) $A \subseteq \operatorname{rk}(X)$ iff for every B such that $B \in A$ there exists Y such that $Y \in X$ and $B \subseteq \operatorname{rk}(Y)$.

(79) $\operatorname{rk}(X) = \emptyset$ iff $X = \emptyset$.

(80) If $\operatorname{rk}(X) = \operatorname{succ} A$, then there exists Y such that $Y \in X$ and $\operatorname{rk}(Y) = A$.

(81) $\operatorname{rk}(A) = A$.

(82) $\operatorname{rk}(\mathbf{T}(X)) \neq \emptyset$ and $\operatorname{rk}(\mathbf{T}(X))$ is a limit ordinal number.

---

[3] The proposition (57) has been removed.
[4] The proposition (70) has been removed.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[8] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[9] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[10] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received March 23, 1990

Published January 2, 2004