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Introduction to Categories and Functors

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Summary. The category is introduced as an ordered 5-tuple of the form $\langle O, M, dom, cod, \cdot, id\rangle$ where O (objects) and M (morphisms) are arbitrary nonempty sets, *dom* and *cod* map M onto O and assign to a morphism domain and codomain, $\cdot$ is a partial binary map from $M \times M$ to M (composition of morphisms), *id* applied to an object yields the identity morphism. We define the basic notions of the category theory such as hom, monic, epi, invertible. We next define functors, the composition of functors, faithfulness and fullness of functors, isomorphism between categories and the identity functor.

MML Identifier: CAT_1.

WWW: http://mizar.org/JFM/Vol1/cat_1.html

The articles [6], [4], [7], [8], [1], [3], [2], and [5] provide the notation and terminology for this paper.

In this paper a, b, c, o, m are sets.

The following proposition is true

(4)[1] Let X, Y, Z be sets, D be a non empty set, and f be a function from X into D. If $Y \subseteq X$ and $f^\circ Y \subseteq Z$, then $f \restriction Y$ is a function from Y into Z.

Let A be a non empty set and let us consider b. Then $A \longmapsto b$ is a function from A into $\{b\}$.

Let us consider a, b, c. The functor $\langle a, b\rangle \longmapsto c$ yields a partial function from $[:\{a\}, \{b\}:]$ to $\{c\}$ and is defined by:

(Def. 1) $\langle a, b\rangle \longmapsto c = \{\langle a, b\rangle\} \longmapsto c$.

The following three propositions are true:

(7)[2] $\mathrm{dom}(\langle a, b\rangle \longmapsto c) = \{\langle a, b\rangle\}$ and $\mathrm{dom}(\langle a, b\rangle \longmapsto c) = [:\{a\}, \{b\}:]$.

(8) $(\langle a, b\rangle \longmapsto c)(\langle a, b\rangle) = c$.

(9) For every element x of $\{a\}$ and for every element y of $\{b\}$ holds $(\langle a, b\rangle \longmapsto c)(\langle x, y\rangle) = c$.

We introduce category structures which are systems

$\langle$ objects, morphisms, a dom-map, a cod-map, a composition, an id-map $\rangle$,

where the objects and the morphisms constitute non empty sets, the dom-map and the cod-map are functions from the morphisms into the objects, the composition is a partial function from [:the morphisms, the morphisms:] to the morphisms, and the id-map is a function from the objects into the morphisms.

[1] The propositions (1)–(3) have been removed.

[2] The propositions (5) and (6) have been removed.

In the sequel C denotes a category structure.

Let us consider C. An object of C is an element of the objects of C. A morphism of C is an element of the morphisms of C.

We adopt the following rules: a, b, c, d denote objects of C and f, g denote morphisms of C.

Let us consider C, f. The functor $\mathrm{dom}\, f$ yields an object of C and is defined as follows:

(Def. 2) $\mathrm{dom}\, f = (\text{the dom-map of } C)(f)$.

The functor $\mathrm{cod}\, f$ yielding an object of C is defined by:

(Def. 3) $\mathrm{cod}\, f = (\text{the cod-map of } C)(f)$.

Let us consider C, f, g. Let us assume that $\langle g, f \rangle \in \mathrm{dom}(\text{the composition of } C)$. The functor $g \cdot f$ yielding a morphism of C is defined by:

(Def. 4) $g \cdot f = (\text{the composition of } C)(\langle g, f \rangle)$.

Let us consider C, a. The functor id_a yielding a morphism of C is defined by:

(Def. 5) $\mathrm{id}_a = (\text{the id-map of } C)(a)$.

Let us consider C, a, b. The functor $\mathrm{hom}(a,b)$ yielding a subset of the morphisms of C is defined as follows:

(Def. 6) $\mathrm{hom}(a,b) = \{f : \mathrm{dom}\, f = a \wedge \mathrm{cod}\, f = b\}$.

The following propositions are true:

(18)[3] $f \in \mathrm{hom}(a,b)$ iff $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$.

(19) $\mathrm{hom}(\mathrm{dom}\, f, \mathrm{cod}\, f) \neq \emptyset$.

Let us consider C, a, b. Let us assume that $\mathrm{hom}(a,b) \neq \emptyset$. A morphism of C is called a morphism from a to b if:

(Def. 7) It $\in \mathrm{hom}(a,b)$.

The following propositions are true:

(21)[4] For every set f such that $f \in \mathrm{hom}(a,b)$ holds f is a morphism from a to b.

(22) Every morphism f of C is a morphism from $\mathrm{dom}\, f$ to $\mathrm{cod}\, f$.

(23) For every morphism f from a to b such that $\mathrm{hom}(a,b) \neq \emptyset$ holds $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$.

(24) Let f be a morphism from a to b and h be a morphism from c to d. If $\mathrm{hom}(a,b) \neq \emptyset$ and $\mathrm{hom}(c,d) \neq \emptyset$ and $f = h$, then $a = c$ and $b = d$.

(25) For every morphism f from a to b such that $\mathrm{hom}(a,b) = \{f\}$ and for every morphism g from a to b holds $f = g$.

(26) For every morphism f from a to b such that $\mathrm{hom}(a,b) \neq \emptyset$ and for every morphism g from a to b holds $f = g$ holds $\mathrm{hom}(a,b) = \{f\}$.

(27) Let f be a morphism from a to b. Suppose $\mathrm{hom}(a,b) \approx \mathrm{hom}(c,d)$ and $\mathrm{hom}(a,b) = \{f\}$. Then there exists a morphism h from c to d such that $\mathrm{hom}(c,d) = \{h\}$.

Let C be a category structure. We say that C is category-like if and only if the conditions (Def. 8) are satisfied.

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[3] The propositions (10)–(17) have been removed.

[4] The proposition (20) has been removed.

(Def. 8)(i) For all elements f, g of the morphisms of C holds $\langle\!\langle g, f\rangle\!\rangle \in \mathrm{dom}$(the composition of C) iff (the dom-map of C)$(g) =$ (the cod-map of C)(f),

(ii) for all elements f, g of the morphisms of C such that (the dom-map of C)$(g) =$ (the cod-map of C)(f) holds (the dom-map of C)((the composition of C)$(\langle\!\langle g, f\rangle\!\rangle)) =$ (the dom-map of C)(f) and (the cod-map of C)((the composition of C)$(\langle\!\langle g, f\rangle\!\rangle)) =$ (the cod-map of C)(g),

(iii) for all elements f, g, h of the morphisms of C such that (the dom-map of C)$(h) =$ (the cod-map of C)(g) and (the dom-map of C)$(g) =$ (the cod-map of C)(f) holds (the composition of C)($\langle\!\langle h$, (the composition of C)$(\langle\!\langle g, f\rangle\!\rangle)\rangle\!\rangle) =$ (the composition of C)($\langle\!\langle$(the composition of C)$(\langle\!\langle h, g\rangle\!\rangle), f\rangle\!\rangle)$, and

(iv) for every element b of the objects of C holds (the dom-map of C)((the id-map of C)$(b)) = b$ and (the cod-map of C)((the id-map of C)$(b)) = b$ and for every element f of the morphisms of C such that (the cod-map of C)$(f) = b$ holds (the composition of C)($\langle\!\langle$(the id-map of C)(b), $f\rangle\!\rangle) = f$ and for every element g of the morphisms of C such that (the dom-map of C)$(g) = b$ holds (the composition of C)($\langle\!\langle g$, (the id-map of C)$(b)\rangle\!\rangle) = g$.

Let us note that there exists a category structure which is category-like.

A category is a category-like category structure.

Let us mention that there exists a category which is strict.

Next we state the proposition

(29)[5] Let C be a category structure. Suppose that

(i) for all morphisms f, g of C holds $\langle\!\langle g, f\rangle\!\rangle \in \mathrm{dom}$(the composition of C) iff $\mathrm{dom}\, g = \mathrm{cod}\, f$,

(ii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(g \cdot f) = \mathrm{dom}\, f$ and $\mathrm{cod}(g \cdot f) = \mathrm{cod}\, g$,

(iii) for all morphisms f, g, h of C such that $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$, and

(iv) for every object b of C holds $\mathrm{dom}(\mathrm{id}_b) = b$ and $\mathrm{cod}(\mathrm{id}_b) = b$ and for every morphism f of C such that $\mathrm{cod}\, f = b$ holds $\mathrm{id}_b \cdot f = f$ and for every morphism g of C such that $\mathrm{dom}\, g = b$ holds $g \cdot \mathrm{id}_b = g$.

Then C is category-like.

Let us consider o, m. The functor $\dot{\circlearrowleft}(o,m)$ yielding a strict category is defined as follows:

(Def. 9) $\dot{\circlearrowleft}(o,m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m,m\rangle \longmapsto m, \{o\} \longmapsto m\rangle$.

The following propositions are true:

(32)[6] o is an object of $\dot{\circlearrowleft}(o,m)$.

(33) m is a morphism of $\dot{\circlearrowleft}(o,m)$.

(34) For every object a of $\dot{\circlearrowleft}(o,m)$ holds $a = o$.

(35) For every morphism f of $\dot{\circlearrowleft}(o,m)$ holds $f = m$.

(36) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ and for every morphism f of $\dot{\circlearrowleft}(o,m)$ holds $f \in \mathrm{hom}(a,b)$.

(37) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ holds every morphism of $\dot{\circlearrowleft}(o,m)$ is a morphism from a to b.

(38) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ holds $\mathrm{hom}(a,b) \neq \emptyset$.

(39) Let a, b, c, d be objects of $\dot{\circlearrowleft}(o,m)$, f be a morphism from a to b, and g be a morphism from c to d. Then $f = g$.

[5] The proposition (28) has been removed.

[6] The propositions (30) and (31) have been removed.

We follow the rules: B, C, D are categories, a, b, c, d are objects of C, and f, f_1, f_2, g, g_1, g_2 are morphisms of C.

We now state several propositions:

(40) $\mathrm{dom}\, g = \mathrm{cod}\, f$ iff $\langle g, f \rangle \in \mathrm{dom}$(the composition of C).

(41) If $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f =$ (the composition of C)$(\langle g, f \rangle)$.

(42) For all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(g \cdot f) = \mathrm{dom}\, f$ and $\mathrm{cod}(g \cdot f) = \mathrm{cod}\, g$.

(43) For all morphisms f, g, h of C such that $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(44) $\mathrm{dom}(\mathrm{id}_b) = b$ and $\mathrm{cod}(\mathrm{id}_b) = b$.

(45) If $\mathrm{id}_a = \mathrm{id}_b$, then $a = b$.

(46) For every morphism f of C such that $\mathrm{cod}\, f = b$ holds $\mathrm{id}_b \cdot f = f$.

(47) For every morphism g of C such that $\mathrm{dom}\, g = b$ holds $g \cdot \mathrm{id}_b = g$.

Let us consider C, g. We say that g is monic if and only if:

(Def. 10) For all f_1, f_2 such that $\mathrm{dom}\, f_1 = \mathrm{dom}\, f_2$ and $\mathrm{cod}\, f_1 = \mathrm{dom}\, g$ and $\mathrm{cod}\, f_2 = \mathrm{dom}\, g$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

Let us consider C, f. We say that f is epi if and only if:

(Def. 11) For all g_1, g_2 such that $\mathrm{dom}\, g_1 = \mathrm{cod}\, f$ and $\mathrm{dom}\, g_2 = \mathrm{cod}\, f$ and $\mathrm{cod}\, g_1 = \mathrm{cod}\, g_2$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Let us consider C, f. We say that f is invertible if and only if:

(Def. 12) There exists g such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $\mathrm{cod}\, g = \mathrm{dom}\, f$ and $f \cdot g = \mathrm{id}_{\mathrm{cod}\, f}$ and $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

In the sequel f denotes a morphism from a to b, f' denotes a morphism from b to a, g denotes a morphism from b to c, and h denotes a morphism from c to d.

One can prove the following two propositions:

(51)[7] If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then $g \cdot f \in \hom(a,c)$.

(52) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then $\hom(a,c) \neq \emptyset$.

Let us consider C, a, b, c, f, g. Let us assume that $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. The functor $g \cdot f$ yields a morphism from a to c and is defined as follows:

(Def. 13) $g \cdot f = g \cdot f$.

We now state three propositions:

(54)[8] If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

(55) $\mathrm{id}_a \in \hom(a,a)$.

(56) $\hom(a,a) \neq \emptyset$.

Let us consider C, a. Then id_a is a morphism from a to a.

We now state a number of propositions:

(57) If $\hom(a,b) \neq \emptyset$, then $\mathrm{id}_b \cdot f = f$.

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[7] The propositions (48)–(50) have been removed.

[8] The proposition (53) has been removed.

(58) If $\hom(b,c) \neq \emptyset$, then $g \cdot \mathrm{id}_b = g$.

(59) $\mathrm{id}_a \cdot \mathrm{id}_a = \mathrm{id}_a$.

(60) Suppose $\hom(b,c) \neq \emptyset$. Then g is monic if and only if for every a and for all morphisms f_1, f_2 from a to b such that $\hom(a,b) \neq \emptyset$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

(61) If $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and g is monic and h is monic, then $h \cdot g$ is monic.

(62) If $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and $h \cdot g$ is monic, then g is monic.

(63) Let h be a morphism from a to b and g be a morphism from b to a. If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$, then g is monic.

(64) id_b is monic.

(65) Suppose $\hom(a,b) \neq \emptyset$. Then f is epi if and only if for every c and for all morphisms g_1, g_2 from b to c such that $\hom(b,c) \neq \emptyset$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

(66) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is epi and g is epi, then $g \cdot f$ is epi.

(67) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $g \cdot f$ is epi, then g is epi.

(68) Let h be a morphism from a to b and g be a morphism from b to a. If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$, then h is epi.

(69) id_b is epi.

(70) Suppose $\hom(a,b) \neq \emptyset$. Then f is invertible if and only if the following conditions are satisfied:

(i) $\hom(b,a) \neq \emptyset$, and

(ii) there exists a morphism g from b to a such that $f \cdot g = \mathrm{id}_b$ and $g \cdot f = \mathrm{id}_a$.

(71) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then for all morphisms g_1, g_2 from b to a such that $f \cdot g_1 = \mathrm{id}_b$ and $g_2 \cdot f = \mathrm{id}_a$ holds $g_1 = g_2$.

Let us consider C, a, b, f. Let us assume that $\hom(a,b) \neq \emptyset$ and f is invertible. The functor f^{-1} yields a morphism from b to a and is defined by:

(Def. 14) $f \cdot f^{-1} = \mathrm{id}_b$ and $f^{-1} \cdot f = \mathrm{id}_a$.

We now state several propositions:

(73)[9] If $\hom(a,b) \neq \emptyset$ and f is invertible, then f is monic and epi.

(74) id_a is invertible.

(75) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $g \cdot f$ is invertible.

(76) If $\hom(a,b) \neq \emptyset$ and f is invertible, then f^{-1} is invertible.

(77) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

Let us consider C, a. We say that a is terminal if and only if:

(Def. 15) $\hom(b,a) \neq \emptyset$ and there exists a morphism f from b to a such that for every morphism g from b to a holds $f = g$.

We say that a is initial if and only if:

[9] The proposition (72) has been removed.

(Def. 16) $\hom(a,b) \neq \emptyset$ and there exists a morphism f from a to b such that for every morphism g from a to b holds $f = g$.

Let us consider b. We say that a and b are isomorphic if and only if:

(Def. 17) $\hom(a,b) \neq \emptyset$ and there exists f which is invertible.

The following propositions are true:

(81)[10] a and b are isomorphic iff $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exist f, f' such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

(82) a is initial iff for every b there exists a morphism f from a to b such that $\hom(a,b) = \{f\}$.

(83) If a is initial, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(84) If a is initial and b is initial, then a and b are isomorphic.

(85) If a is initial and a and b are isomorphic, then b is initial.

(86) b is terminal iff for every a there exists a morphism f from a to b such that $\hom(a,b) = \{f\}$.

(87) If a is terminal, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(88) If a is terminal and b is terminal, then a and b are isomorphic.

(89) If b is terminal and a and b are isomorphic, then a is terminal.

(90) If $\hom(a,b) \neq \emptyset$ and a is terminal, then f is monic.

(91) a and a are isomorphic.

(92) If a and b are isomorphic, then b and a are isomorphic.

(93) If a and b are isomorphic and b and c are isomorphic, then a and c are isomorphic.

Let us consider C, D. A function from the morphisms of C into the morphisms of D is said to be a functor from C to D if it satisfies the conditions (Def. 18).

(Def. 18)(i) For every element c of the objects of C there exists an element d of the objects of D such that $\mathrm{it}((\text{the id-map of } C)(c)) = (\text{the id-map of } D)(d)$,

(ii) for every element f of the morphisms of C holds $\mathrm{it}((\text{the id-map of } C)((\text{the dom-map of } C)(f))) = (\text{the id-map of } D)((\text{the dom-map of } D)(\mathrm{it}(f)))$ and $\mathrm{it}((\text{the id-map of } C)((\text{the cod-map of } C)(f))) = (\text{the id-map of } D)((\text{the cod-map of } D)(\mathrm{it}(f)))$, and

(iii) for all elements f, g of the morphisms of C such that $\langle g, f\rangle \in \mathrm{dom}(\text{the composition of } C)$ holds $\mathrm{it}((\text{the composition of } C)(\langle g, f\rangle)) = (\text{the composition of } D)(\langle \mathrm{it}(g), \mathrm{it}(f)\rangle)$.

The following propositions are true:

(96)[11] Let T be a function from the morphisms of C into the morphisms of D. Suppose that

(i) for every object c of C there exists an object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom} f}) = \mathrm{id}_{\mathrm{dom} T(f)}$ and $T(\mathrm{id}_{\mathrm{cod} f}) = \mathrm{id}_{\mathrm{cod} T(f)}$, and

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.

Then T is a functor from C to D.

(97) For every functor T from C to D and for every object c of C there exists an object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$.

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[10] The propositions (78)–(80) have been removed.

[11] The propositions (94) and (95) have been removed.

(98) For every functor T from C to D and for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom}\,f}) = \mathrm{id}_{\mathrm{dom}\,T(f)}$ and $T(\mathrm{id}_{\mathrm{cod}\,f}) = \mathrm{id}_{\mathrm{cod}\,T(f)}$.

(99) Let T be a functor from C to D and f, g be morphisms of C. If $\mathrm{dom}\,g = \mathrm{cod}\,f$, then $\mathrm{dom}\,T(g) = \mathrm{cod}\,T(f)$ and $T(g \cdot f) = T(g) \cdot T(f)$.

(100) Let T be a function from the morphisms of C into the morphisms of D and F be a function from the objects of C into the objects of D. Suppose that

(i) for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{F(c)}$,

(ii) for every morphism f of C holds $F(\mathrm{dom}\,f) = \mathrm{dom}\,T(f)$ and $F(\mathrm{cod}\,f) = \mathrm{cod}\,T(f)$, and

(iii) for all morphisms f, g of C such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.

Then T is a functor from C to D.

Let us consider C, D and let F be a function from the morphisms of C into the morphisms of D. Let us assume that for every element c of the objects of C there exists an element d of the objects of D such that $F(($the id-map of $C)(c)) = ($the id-map of $D)(d)$. The functor $\mathrm{Obj}\,F$ yields a function from the objects of C into the objects of D and is defined by the condition (Def. 19).

(Def. 19) Let c be an element of the objects of C and d be an element of the objects of D. Suppose $F(($the id-map of $C)(c)) = ($the id-map of $D)(d)$. Then $(\mathrm{Obj}\,F)(c) = d$.

The following four propositions are true:

(102)[12] Let T be a function from the morphisms of C into the morphisms of D. Suppose that for every object c of C there exists an object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$. Let c be an object of C and d be an object of D. If $T(\mathrm{id}_c) = \mathrm{id}_d$, then $(\mathrm{Obj}\,T)(c) = d$.

(103) Let T be a functor from C to D, c be an object of C, and d be an object of D. If $T(\mathrm{id}_c) = \mathrm{id}_d$, then $(\mathrm{Obj}\,T)(c) = d$.

(104) For every functor T from C to D and for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{(\mathrm{Obj}\,T)(c)}$.

(105) For every functor T from C to D and for every morphism f of C holds $(\mathrm{Obj}\,T)(\mathrm{dom}\,f) = \mathrm{dom}\,T(f)$ and $(\mathrm{Obj}\,T)(\mathrm{cod}\,f) = \mathrm{cod}\,T(f)$.

Let C, D be categories, let T be a functor from C to D, and let c be an object of C. The functor $T(c)$ yields an object of D and is defined by:

(Def. 20) $T(c) = (\mathrm{Obj}\,T)(c)$.

One can prove the following four propositions:

(107)[13] Let T be a functor from C to D, c be an object of C, and d be an object of D. If $T(\mathrm{id}_c) = \mathrm{id}_d$, then $T(c) = d$.

(108) For every functor T from C to D and for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{T(c)}$.

(109) For every functor T from C to D and for every morphism f of C holds $T(\mathrm{dom}\,f) = \mathrm{dom}\,T(f)$ and $T(\mathrm{cod}\,f) = \mathrm{cod}\,T(f)$.

(110) For every functor T from B to C and for every functor S from C to D holds $S \cdot T$ is a functor from B to D.

Let us consider B, C, D, let T be a functor from B to C, and let S be a functor from C to D. Then $S \cdot T$ is a functor from B to D.

The following three propositions are true:

(111) $\mathrm{id}_{\text{the morphisms of } C}$ is a functor from C to C.

[12] The proposition (101) has been removed.

[13] The proposition (106) has been removed.

(112) Let T be a functor from B to C, S be a functor from C to D, and b be an object of B. Then $(\mathrm{Obj}(S \cdot T))(b) = (\mathrm{Obj}\,S)((\mathrm{Obj}\,T)(b))$.

(113) For every functor T from B to C and for every functor S from C to D and for every object b of B holds $(S \cdot T)(b) = S(T(b))$.

Let us consider C. The functor id_C yields a functor from C to C and is defined as follows:

(Def. 21) $\mathrm{id}_C = \mathrm{id}_{\text{the morphisms of } C}$.

The following four propositions are true:

(115)[14] For every morphism f of C holds $\mathrm{id}_C(f) = f$.

(116) For every object c of C holds $(\mathrm{Obj}(\mathrm{id}_C))(c) = c$.

(117) $\mathrm{Obj}(\mathrm{id}_C) = \mathrm{id}_{\text{the objects of } C}$.

(118) For every object c of C holds $\mathrm{id}_C(c) = c$.

Let C, D be categories and let T be a functor from C to D. We say that T is isomorphic if and only if:

(Def. 22) T is one-to-one and $\mathrm{rng}\,T =$ the morphisms of D and $\mathrm{rng}\,\mathrm{Obj}\,T =$ the objects of D.

We introduce T is an isomorphism as a synonym of T is isomorphic. We say that T is full if and only if the condition (Def. 23) is satisfied.

(Def. 23) Let c, c' be objects of C. Suppose $\hom(T(c), T(c')) \neq \emptyset$. Let g be a morphism from $T(c)$ to $T(c')$. Then $\hom(c, c') \neq \emptyset$ and there exists a morphism f from c to c' such that $g = T(f)$.

We say that T is faithful if and only if:

(Def. 24) For all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ and for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

One can prove the following propositions:

(122)[15] id_C is an isomorphism.

(123) Let T be a functor from C to D, c, c' be objects of C, and f be a set. If $f \in \hom(c, c')$, then $T(f) \in \hom(T(c), T(c'))$.

(124) Let T be a functor from C to D and c, c' be objects of C. If $\hom(c, c') \neq \emptyset$, then for every morphism f from c to c' holds $T(f) \in \hom(T(c), T(c'))$.

(125) Let T be a functor from C to D and c, c' be objects of C. Suppose $\hom(c, c') \neq \emptyset$. Let f be a morphism from c to c'. Then $T(f)$ is a morphism from $T(c)$ to $T(c')$.

(126) For every functor T from C to D and for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ holds $\hom(T(c), T(c')) \neq \emptyset$.

(127) Let T be a functor from B to C and S be a functor from C to D. If T is full and S is full, then $S \cdot T$ is full.

(128) Let T be a functor from B to C and S be a functor from C to D. If T is faithful and S is faithful, then $S \cdot T$ is faithful.

(129) For every functor T from C to D and for all objects c, c' of C holds $T^\circ \hom(c, c') \subseteq \hom(T(c), T(c'))$.

[14] The proposition (114) has been removed.

[15] The propositions (119)–(121) have been removed.

Let C, D be categories, let T be a functor from C to D, and let c, c' be objects of C. The functor $T_{c,c'}$ yielding a function from $\hom(c,c')$ into $\hom(T(c),T(c'))$ is defined by:

(Def. 25) $T_{c,c'} = T \restriction \hom(c,c')$.

One can prove the following propositions:

(131)[16] Let T be a functor from C to D and c, c' be objects of C. If $\hom(c,c') \neq \emptyset$, then for every morphism f from c to c' holds $T_{c,c'}(f) = T(f)$.

(132) For every functor T from C to D holds T is full iff for all objects c, c' of C holds $\mathrm{rng}(T_{c,c'}) = \hom(T(c),T(c'))$.

(133) Let T be a functor from C to D. Then T is faithful if and only if for all objects c, c' of C holds $T_{c,c'}$ is one-to-one.

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[16] The proposition (130) has been removed.